Annexe C

Expressions des différentes fonctions potentiels

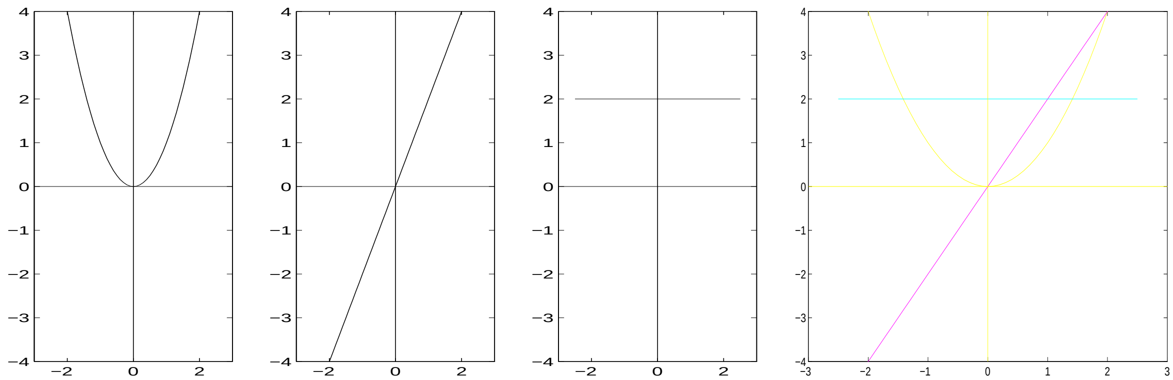
Dans cette annexe, nous faisons un inventaires des fonctions potentiels utilisées en modélisation markovienne pour la résolution des problèmes inverses.

Ces fonctions sont classées en deux familles : convexes et non convexes.

C.1 Potentiels convexes

Quadratique ou L_2:

$$\begin{aligned}\phi(t) &= t^2\\ \phi'(t) &= 2t\\ \phi''(t) &= 2\end{aligned}$$

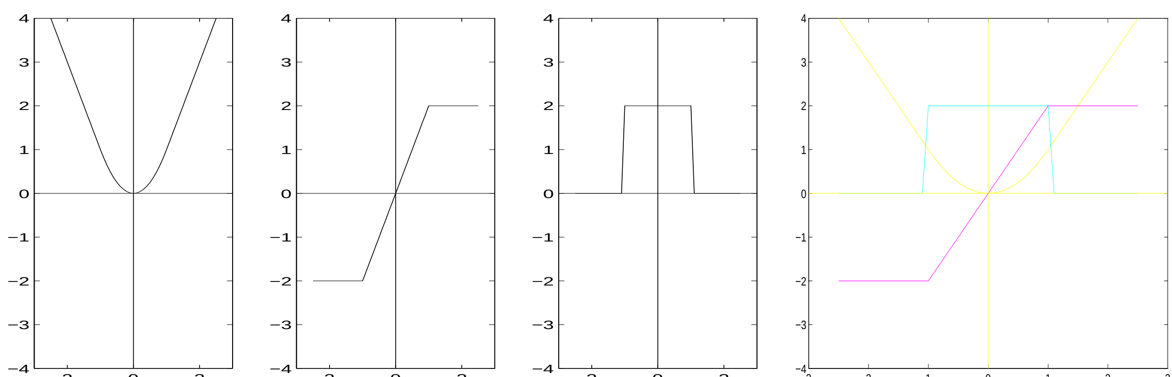


Quadratique prolongée en linéaire :

$$\phi(t) = \begin{cases} t^2 & |t| \leq 1 \\ 2t-1 & |t| > 1 \end{cases}$$

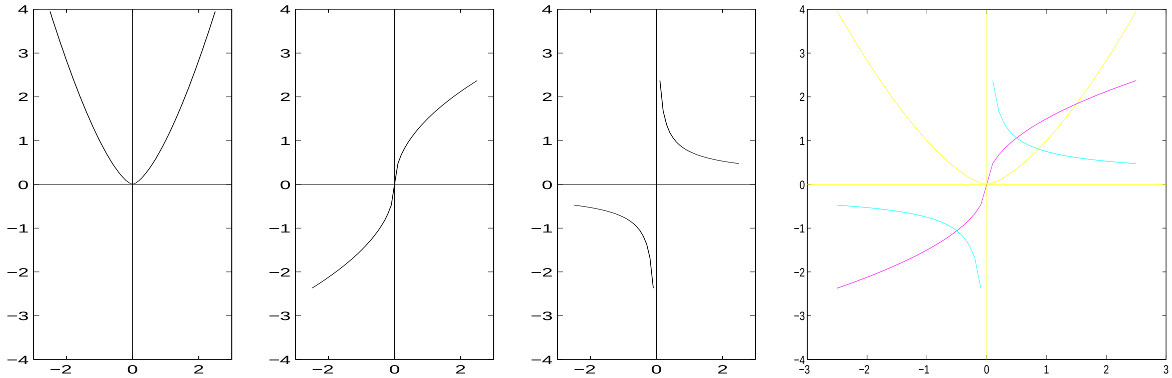
$$\phi'(t) = \begin{cases} 2t & |t| \leq 1 \\ 2 & |t| > 1 \end{cases}$$

$$\phi''(t) = \begin{cases} 2 & |t| \leq 1 \\ 0 & |t| > 1 \end{cases}$$



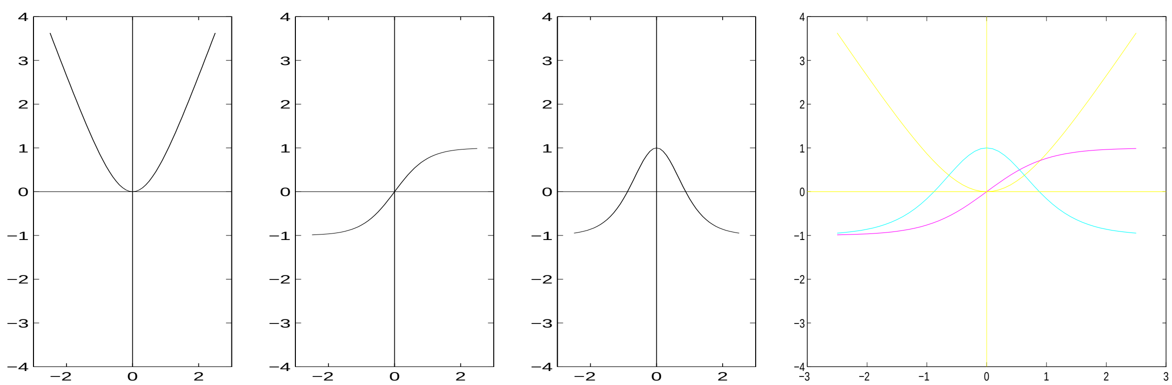
L_p :

$$
\begin{aligned}
\phi(t) &= |t|^p, \quad 1 < p \leq 2 \\
\phi'(t) &= p\,\mathrm{sign}(t)|t|^{p-1} \\
\phi''(t) &= p(p-1)\,|t|^{p-2}
\end{aligned}
$$



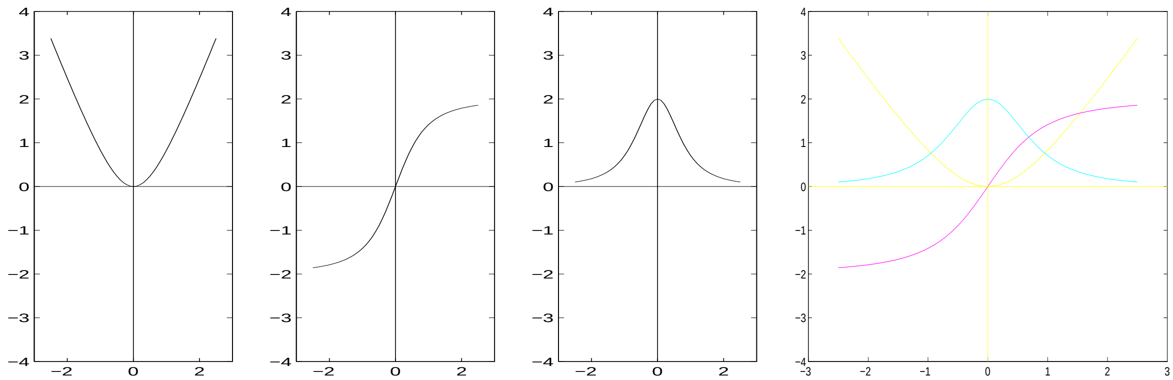
log cosh :

$$
\begin{aligned}
\phi(t) &= 2\log(\cosh(t)) \\
\phi'(t) &= \frac{\sinh(t)}{\cosh(t)} \\
\phi''(t) &= \frac{1-\sinh^2(t)}{\cosh^2(t)}
\end{aligned}
$$



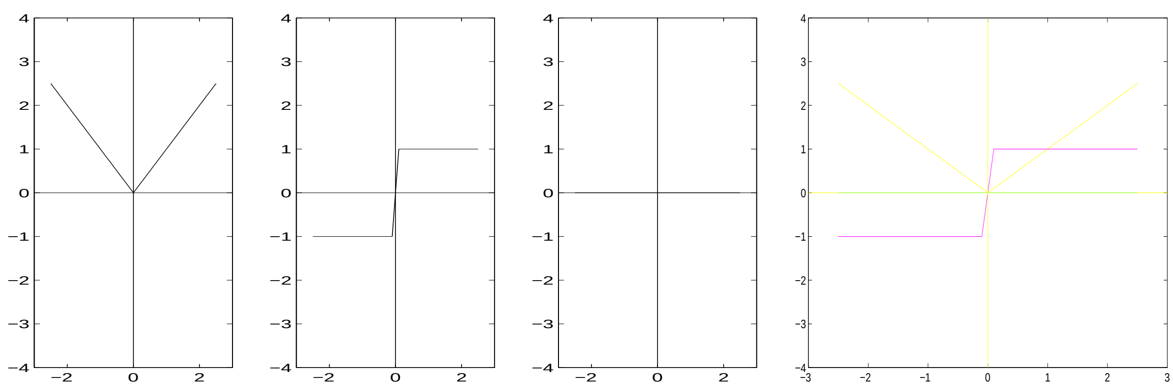
Racine carrée :

$$\begin{aligned}\phi(t) &= 2\sqrt{1+t^2}-2\\ \phi'(t) &= \frac{2t}{\sqrt{1+t^2}}\\ \phi''(t) &= \frac{2}{(1+t^2)^{3/2}}\end{aligned}$$



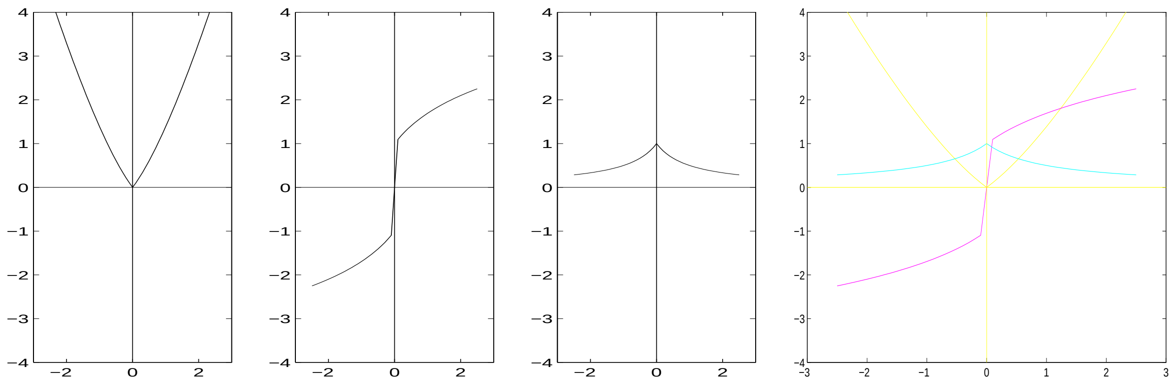
L_1 **:**

$$\begin{aligned}\phi(t) &= |t|\\ \phi'(t) &= \text{sign}(t)\\ \phi''(t) &= 0\end{aligned}$$



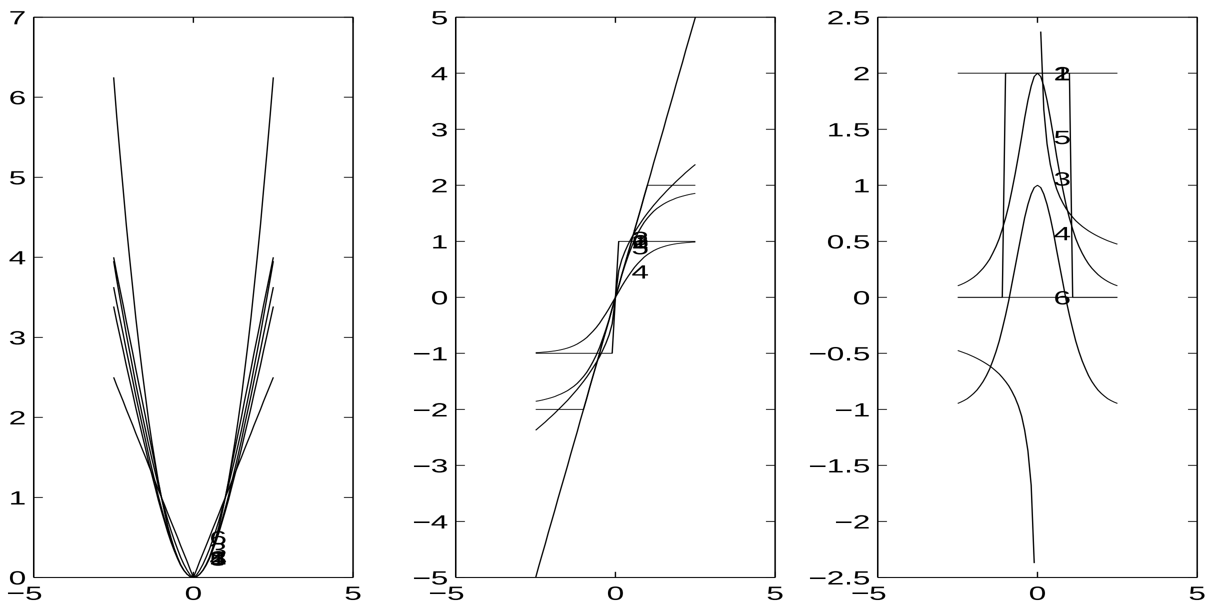
Logarithmique :

$$\begin{aligned}\phi(t) &= (|t|+1)\log(|t|+1)\\ \phi'(t) &= \mathrm{sign}(t)\,(1+\log(|t|+1))\\ \phi''(t) &= \frac{1}{|t|+1}\end{aligned}$$



Le tableau ci-dessous résume ces fonctions :

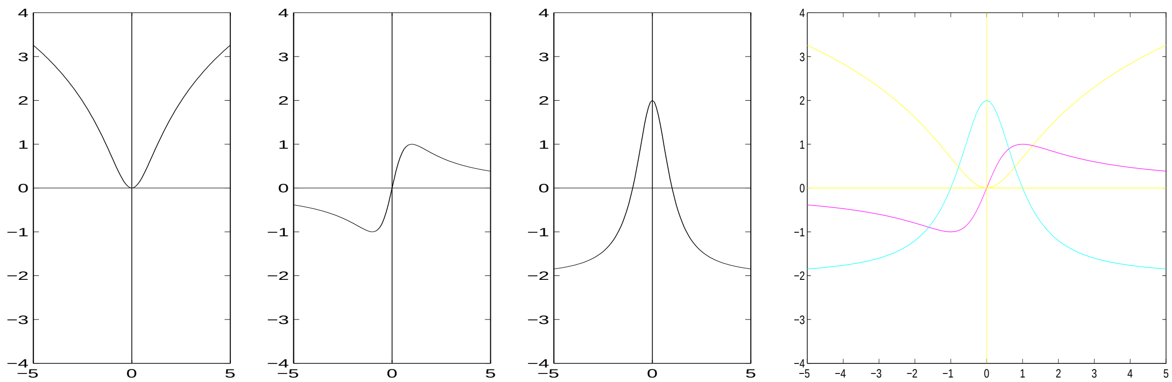
cas	$\phi(t)$	$\phi'(t)$	$\phi''(t)$	références
1	t^2	$2t$	2	Tikhonov
2	$\begin{cases} t^2 & \lvert t\rvert \le 1 \\ 2t-1 & \lvert t\rvert > 1 \end{cases}$	$\begin{cases} 2t & \lvert t\rvert \le 1 \\ 2 & \lvert t\rvert > 1 \end{cases}$	$\begin{cases} 2 & \lvert t\rvert \le 1 \\ 0 & \lvert t\rvert > 1 \end{cases}$	Hubert 81
3	$\lvert t\rvert^p, \quad 1 < p \le 2$	$p\,\mathrm{sign}(t)\lvert t\rvert^{p-1}$	$p(p-1)\,\lvert t\rvert^{p-2}$	Bouman & Sauer 93
4	$2\log(\cosh(t))$	$\frac{\sinh(t)}{\cosh(t)}$	$\frac{1-\sinh^2(t)}{\cosh^2(t)}$	Green 90
5	$2\sqrt{1+t^2}-2$	$\frac{2t}{\sqrt{1+t^2}}$	$\frac{2}{(1+t^2)^{3/2}}$	Aubert 94
6	$\lvert t\rvert$	$\mathrm{sign}(t)$	0	Rudin Osher 92, Besag 86
7	$(\lvert t\rvert+1)\log(\lvert t\rvert+1)$	$\mathrm{sign}(t)\,(1+\log(\lvert t\rvert+1))$	$\frac{1}{\lvert t\rvert+1}$	Zervakis 95
8	$(\lvert t-m\rvert+\lvert t\rvert+1)$ $\log\left(\frac{\lvert t-m\rvert+\lvert t\rvert+1}{\lvert t\rvert+1}+1\right)$	$\mathrm{sign}(t-m)$ $\log\left(\frac{\lvert t-m\rvert}{\lvert t\rvert+1}+1\right)$	$\frac{1}{\lvert t-m\rvert+\lvert t\rvert+1}$	Zervakis 95
9	$\begin{cases} t^2 & \lvert t\rvert \le 1 \\ 1-2\log\lvert t\rvert & \lvert t\rvert > 1 \end{cases}$	$\begin{cases} 2t & \lvert t\rvert \le 1 \\ \frac{2\lvert t\rvert}{t^2} & \lvert t\rvert > 1 \end{cases}$	$\begin{cases} 2 & \lvert t\rvert \le 1 \\ \frac{2}{t^2} & \lvert t\rvert > 1 \end{cases}$	AMD



C.2 Potentiels non convexes

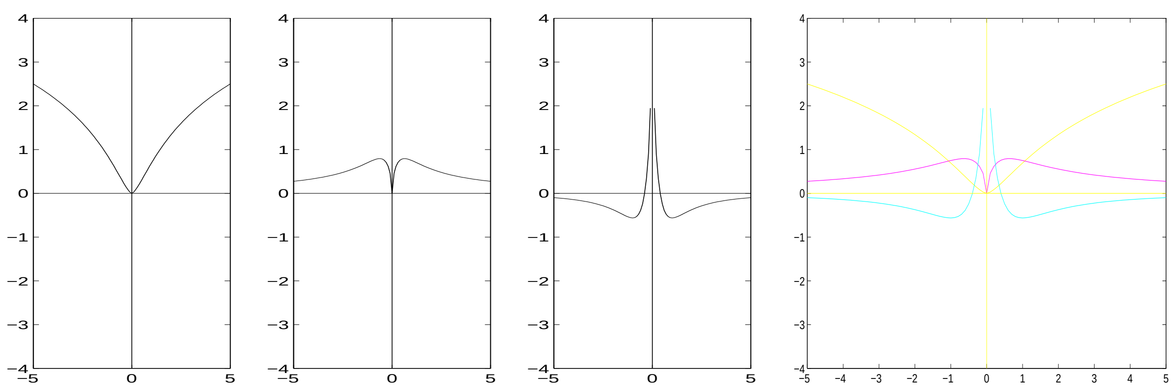
Hebert & Leahy, Perona & Malik :

$$\begin{aligned} \phi(t) &= \log(1+t^2) \\ \phi'(t) &= \frac{2t}{1+t^2} \\ \phi''(t) &= \frac{2(1-t^2)}{(1+t^2)^2} \end{aligned}$$



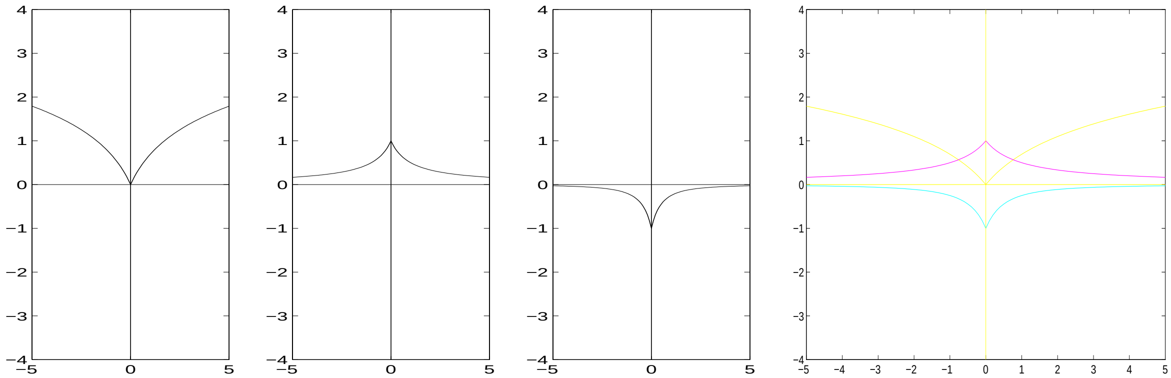
Extension :

$$\begin{aligned} \phi(t) &= \log(1+|t|^p) \\ \phi'(t) &= \frac{p\,|t|^{p-1}}{1+t^p} \\ \phi''(t) &= \frac{p(p-1)|t|^{p-2}(1-t^p) - p^2 t^{2p-2}}{(1+t^p)^2} \end{aligned}$$



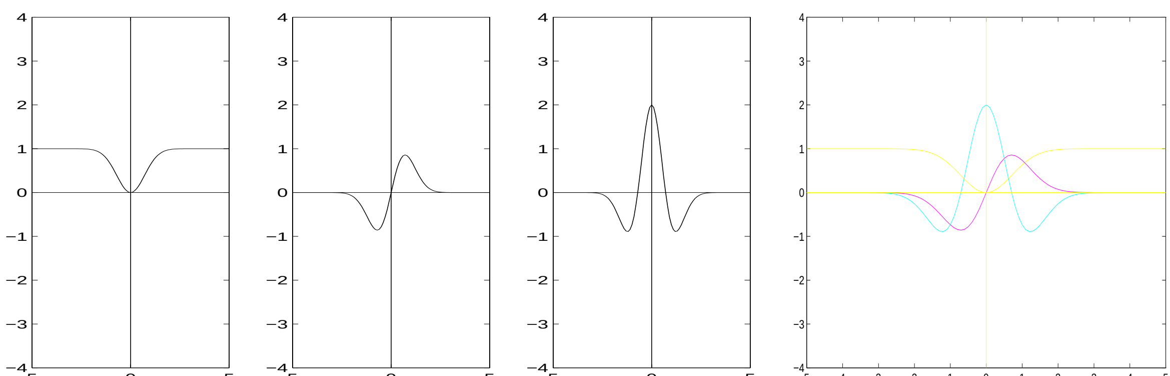
Cas particulier $p = 1$:

$$
\begin{aligned}
\phi(t) &= \log(1+|t|) \\
\phi'(t) &= \frac{\operatorname{sign}(t)}{1+t} \\
\phi''(t) &= \frac{-1}{(1+t)^2}
\end{aligned}
$$



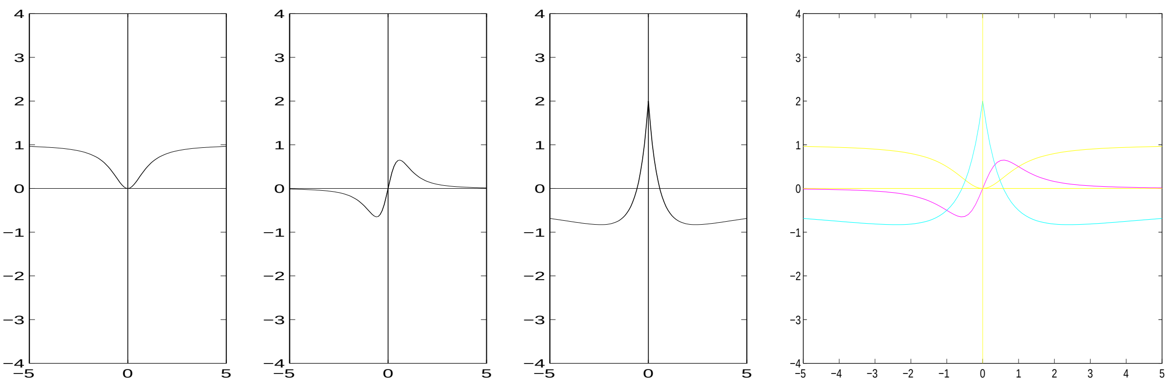
Gaussienne inversée :

$$
\begin{aligned}
\phi(t) &= 1 - e^{-t^2} \\
\phi'(t) &= 2te^{-t^2} \\
\phi''(t) &= 2e^{-t^2}(1-2t^2)
\end{aligned}
$$



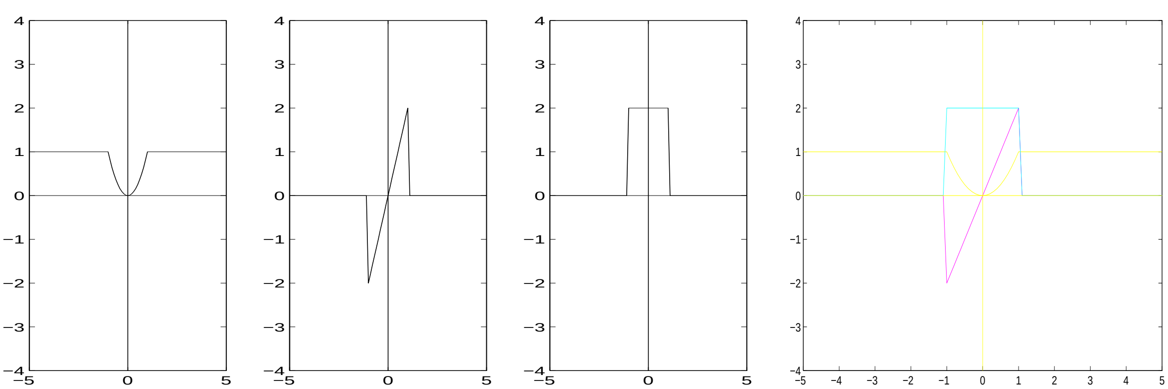
Lorentzienne inversée :

$$\phi(t) = \frac{t^2}{1+t^2}$$
$$\phi'(t) = \frac{2t}{(1+t^2)^2}$$
$$\phi''(t) = \frac{2(1-3t^2)}{(1+t)^3}$$



Quadratique tronquée :

$$\phi(t) = \begin{cases} 2t & |t| \leq 1 \\ 0 & |t| > 1 \end{cases}$$
$$\phi'(t) = \begin{cases} 2t & |t| \leq 1 \\ 0 & |t| > 1 \end{cases}$$
$$\phi''(t) = \begin{cases} 2 & |t| \leq 1 \\ 0 & |t| > 1 \end{cases}$$



Le tableau ci-dessous résume ces fonctions :

cas	$\phi(t)$	$\phi'(t)$	$\phi''(t)$	références
1	$\log(1+t^2)$	$\frac{2t}{1+t^2}$	$\frac{2(1-t^2)}{(1+t^2)^2}$	Hebert & Leahy 90
				Perona & Malik 90
2	$\log(1+\|t\|^p)$	$\frac{p\,\|t\|^{p-1}}{1+t^p}$	$\frac{p(p-1)\|t\|^{p-2}(1-t^p)-p^2\,t^{2p-2}}{(1+t^p)^2}$	AMD
3	$\log(1+\|t\|)$	$\mathrm{sign}(t)\frac{1}{1+t}$	$\frac{-1}{(1+t)^2}$	AMD
4	$-e^{-t^2}+1$	$2te^{-t^2}$	$2e^{-t^2}(1-2t^2)$	Perona & Malik 90
5	$\frac{t^2}{1+t^2}$	$\frac{2t}{(1+t^2)^2}$	$\frac{2(1-3t^2)}{(1+t)^3}$	Geman & McClure 85 90
6	$\min(1,t^2) = \begin{cases} t^2 & \|t\| \leq 1 \\ 1 & \|t\| > 1 \end{cases}$	$\begin{cases} 2t & \|t\| \leq 1 \\ 0 & \|t\| > 1 \end{cases}$	$\begin{cases} 2 & \|t\| \leq 1 \\ 0 & \|t\| > 1 \end{cases}$	Blake & Zisserman 87
				Geman & Geman 84