Annexe D

Quelques exemples simples d'inférence bayésienne

Dans cette annexe, quelques exemples simples de l'inférence bayésienne sont présentés. L'objective étant plutôt pédagogique.

D.1 Introduction et rappel des notations

Dans ces exemples nous utilisons les notations suivantes :

$$N(m,\lambda) = \sqrt{\frac{\lambda}{2\pi}} \exp\left[-\frac{\lambda}{2}(x-m)^2\right]$$

$$N(\boldsymbol{m},\boldsymbol{\Lambda}) = \frac{|\boldsymbol{\Lambda}|^{1/2}}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{m})^t\boldsymbol{\Lambda}(\boldsymbol{x}-\boldsymbol{m})\right]$$

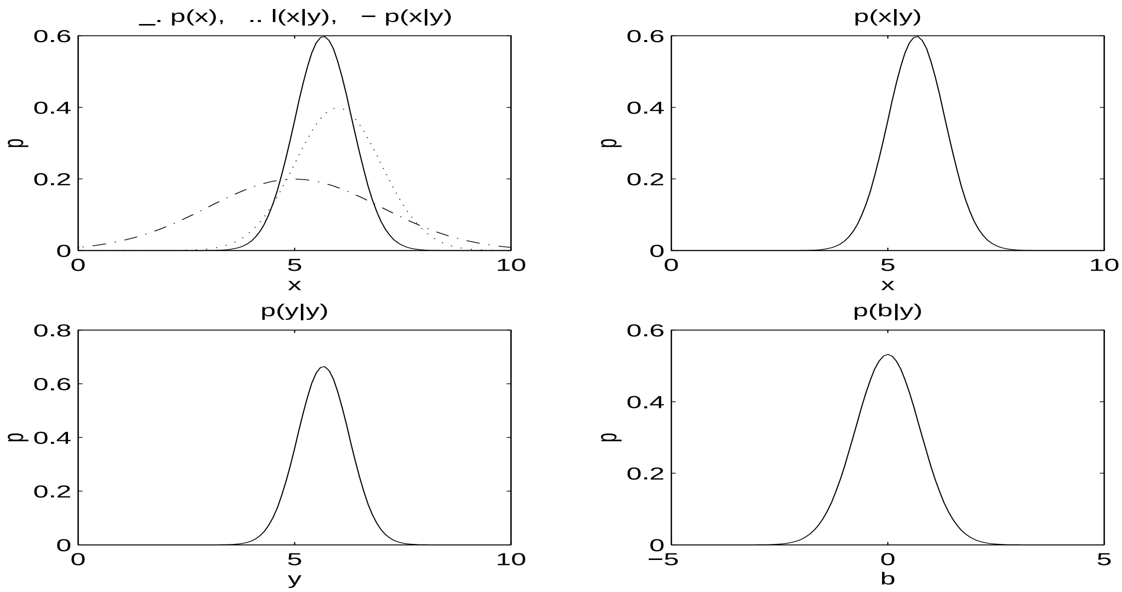
Problème 1 :

$$y = x + b$$

$$\begin{array}{ll} b|\beta & \sim \mathbf{N}(0,\beta) \\ y|x,\beta & \sim \mathbf{N}(x,\beta) \\ & \\ x|x_0,\theta & \sim \mathbf{N}(x_0,\theta) \\ & \\ \begin{pmatrix} y \\ x \end{pmatrix} |\beta, x_0, \theta & \sim \mathbf{N}\left(\begin{pmatrix} x_0 \\ x_0 \end{pmatrix}, \begin{pmatrix} \theta+\beta & \theta \\ \theta & \theta \end{pmatrix}^{-1}\right) \\ & \\ y|\beta,x_0,\theta & \sim \mathbf{N}\left(x_0, \frac{\theta\beta}{\theta+\beta}\right) \\ & \\ x|y,\beta,x_0,\theta & \sim \mathbf{N}(x_m,\theta_m), \\ & \text{avec} \\ & x_m = \frac{1}{\theta_m}(\theta x_0 + \beta y) \\ & \theta_m = \theta+\beta \\ & \\ y|y,\beta,x_0,\theta & \sim \mathbf{N}\left(x_m, \frac{\theta_m\beta}{\theta_m+\beta}\right) \\ b|y,\beta,x_0,\theta & \sim \mathbf{N}\left(0, \frac{2\theta_m\beta}{2\theta_m+\beta}\right) \end{array}$$

$$\begin{array}{llll} \beta = 0 & \longrightarrow & x_m = x_0, & \theta_m = \theta \\ \theta = 0 & \longrightarrow & x_m = y, & \theta_m = \beta \\ \beta \longrightarrow \infty & \longrightarrow & x_m = y, & \theta_m = \infty \\ \theta \longrightarrow \infty & \longrightarrow & x_m = x_0, & \theta_m = \infty \\ & & & \\ \theta = r\beta & \longrightarrow & x_m = \frac{1}{r+1}(rx_0 + y), & \theta_m = (r+1)\beta \end{array}$$

Exemple 1 : $y = 6, \quad \beta = 1, \quad x_0 = 5, \quad \theta = .5 \longrightarrow x_m = 5.6666, \quad \theta_m = 1.5$



Problème 2 :

$$y_i = x + b_i, \qquad i = 1, \cdots, m \quad \longrightarrow \boldsymbol{y} = x\mathbf{1} + \boldsymbol{b}$$

$$\begin{array}{ll} b_i|\beta & \sim \mathbf{N}(0, \beta) \\ \boldsymbol{b}|\beta & \sim \mathbf{N}(\mathbf{0}, \beta \boldsymbol{I}) \\ y_i|x, \beta & \sim \mathbf{N}(x, \beta) \\ \boldsymbol{y}|x, \beta & \sim \mathbf{N}(x\mathbf{1}, \beta \boldsymbol{I}) \end{array}$$

$$x|x_0, \theta \qquad \sim \mathbf{N}(x_0, \theta)$$

$$\begin{array}{ll} \begin{pmatrix} y_i \\ x \end{pmatrix} |\beta, x_0, \theta & \sim \mathbf{N}\left(\begin{pmatrix} x_0 \\ x_0 \end{pmatrix}, \begin{pmatrix} \theta + \beta & \theta \\ \theta & \theta \end{pmatrix}^{-1}\right) \\ \begin{pmatrix} \boldsymbol{y} \\ x \end{pmatrix} |\beta, x_0, \theta & \sim \mathbf{N}\left(\begin{pmatrix} x_0\mathbf{1} \\ x_0 \end{pmatrix}, \begin{pmatrix} (\theta + \beta)\boldsymbol{I} & \theta\mathbf{1} \\ \theta\mathbf{1}^t & \theta \end{pmatrix}^{-1}\right) \end{array}$$

$$\begin{array}{ll} y_i|\beta, x_0, \theta & \sim \mathbf{N}(x_0, \frac{\theta\beta}{\theta+\beta}) \\ \boldsymbol{y}|\beta, x_0, \theta & \sim \mathbf{N}(x_0\mathbf{1}, \frac{\theta\beta}{\theta+\beta}\boldsymbol{I}) \end{array}$$

$$x|\boldsymbol{y}, \beta, x_0, \theta \qquad \sim \mathbf{N}(x_m, \theta_m),$$

avec

$$x_m = \frac{1}{\theta_m}(\theta x_0 + m\beta\bar{y}) \longrightarrow x_m = \frac{1}{\theta_m}(\theta_{m-1}x_{m-1} + \beta y_m)$$

$$\theta_m = \theta + m\beta \longrightarrow \qquad \theta_m = \theta_{m-1} + \beta$$

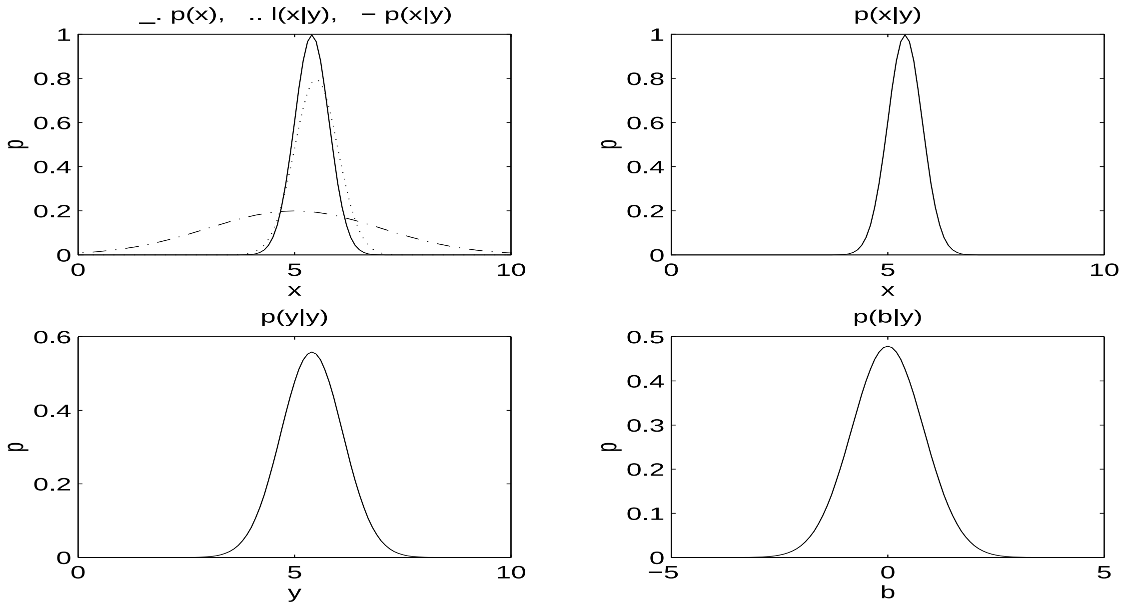
$$\bar{y} = \frac{1}{m}\sum_i y_i$$

$$\begin{array}{ll} y_k|\boldsymbol{y}, \beta, x_0, \theta & \sim \mathbf{N}\left(x_m, \frac{\theta_m\beta}{\theta_m+\beta}\right) \\ b_k|\boldsymbol{y}, \beta, x_0, \theta & \sim \mathbf{N}\left(0, \frac{2\theta_m\beta}{2\theta_m+\beta}\right) \end{array}$$

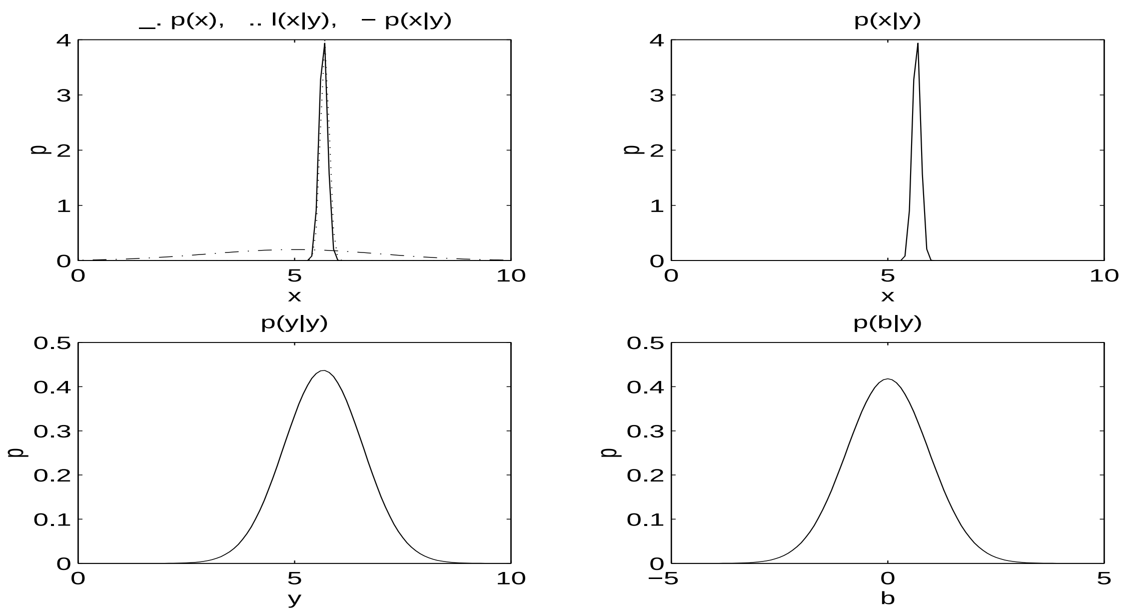
$$\begin{array}{llll} \beta = 0 & \longrightarrow & x_m = x_0, & \theta_m = \theta \\ \theta = 0 & \longrightarrow & x_m = \bar{y}, & \theta_m = m\beta \\ \beta \longrightarrow \infty & \longrightarrow & x_m = \bar{y}, & \theta_m = \infty \\ \theta \longrightarrow \infty & \longrightarrow & x_m = x_0, & \theta_m = \infty \\ m \longrightarrow \infty & \longrightarrow & x_m = \bar{y}, & \theta_m = \infty \end{array}$$

$$\theta = r\beta \quad \longrightarrow \quad x_m = \frac{1}{r+m}(rx_0 + m\bar{y}), \quad \theta_m = (r+m)\beta$$

Exemple 2.1 : $\boldsymbol{y} = [6, 5], \quad m = 2, \quad \bar{y} = 5.5, \quad \beta = 1, \quad x_0 = 5, \quad \theta = .5$
$x_m = 5.4, \quad \theta_m = 2.5$



Exemple 2.2 : $\boldsymbol{y} = [6, 5, 4, 7, 3, 5, 6, 8, 7, 6], \quad m = 10, \quad \bar{y} = 5.7$
$\beta = 1, \quad x_0 = 5, \quad \theta = .5 \; x_m = 5.667, \quad \theta_m = 10.5$



Problème 3 :

$$y = ax + b$$

$$\begin{array}{ll} b|\beta & \sim \mathbf{N}(0,\beta) \\ y|x,\beta & \sim \mathbf{N}(ax,\beta) \end{array}$$

$$x|x_0,\theta \qquad \sim \mathbf{N}(x_0,\theta)$$

$$\begin{pmatrix} y \\ x \end{pmatrix} |\beta, x_0, \theta \quad \sim \mathbf{N}\left(\begin{pmatrix} ax_0 \\ x_0 \end{pmatrix}, \begin{pmatrix} \theta + a^2\beta & a\theta \\ a\theta & \theta \end{pmatrix}^{-1} \right)$$

$$y|\beta, x_0, \theta \quad \sim \mathbf{N}\left(ax_0, \frac{\theta\beta}{\theta + a^2\beta}\right)$$

$$\begin{array}{ll} x|y,\beta,x_0,\theta & \sim \mathbf{N}(x_m,\theta_m), \\ & \text{avec} \\ & x_m = \frac{1}{\theta_m}(\theta x_0 + a\beta y), \\ & \theta_m = \theta + a^2\beta \end{array}$$

$$\begin{array}{ll} y|y,\beta,x_0,\theta & \sim \mathbf{N}\left(ax_m, \frac{\theta_m\beta}{\theta_m + a^2\beta}\right) \\ b|y,\beta,x_0,\theta & \sim \mathbf{N}\left(0, \frac{2\theta_m\beta}{2\theta_m + a^2\beta}\right) \end{array}$$

$$\begin{array}{llll} \beta = 0 & \longrightarrow & x_m = x_0, & \theta_m = \theta \\ \theta = 0 & \longrightarrow & x_m = y/a, & \theta_m = a^2\beta \\ \beta \longrightarrow \infty & \longrightarrow & x_m = y/a, & \theta_m = \infty \\ \theta \longrightarrow \infty & \longrightarrow & x_m = x_0, & \theta_m = \infty \end{array}$$

$$\begin{array}{llll} \theta = r\beta & \longrightarrow & x_m = \frac{1}{r+a^2}(rx_0 + ay), & \theta_m = (r + a^2)\beta \\ r \longrightarrow \infty & \longrightarrow & x_m = x_0, & \theta_m = \infty \\ r = 0 & \longrightarrow & x_m = y/a, & \theta_m = a^2\beta \end{array}$$

Problème 4 :

$$y_i = ax + b_i, \quad i = 1, \cdots, m \quad \longrightarrow \boldsymbol{y} = ax\mathbf{1} + \boldsymbol{b}$$

$$\begin{array}{ll}
b_i|\beta & \sim \mathbf{N}(0, \beta) \\
\boldsymbol{b}|\beta & \sim \mathbf{N}(\mathbf{0}, \beta \boldsymbol{I}) \\
y_i|x, \beta & \sim \mathbf{N}(ax, \beta) \\
\boldsymbol{y}|x, \beta & \sim \mathbf{N}(ax\mathbf{1}, \beta \boldsymbol{I})
\end{array}$$

$$x|x_0, \theta \sim \mathbf{N}(x_0, \theta)$$

$$\begin{pmatrix} y_i \\ x \end{pmatrix} |\beta, x_0, \theta \sim \mathbf{N}\left(\begin{pmatrix} ax_0 \\ x_0 \end{pmatrix}, \begin{pmatrix} a^2\theta + \beta & a\theta \\ a\theta & \theta \end{pmatrix}^{-1} \right)$$

$$\begin{pmatrix} \boldsymbol{y} \\ x \end{pmatrix} |\beta, x_0, \theta \sim \mathbf{N}\left(\begin{pmatrix} ax_0\mathbf{1} \\ x_0 \end{pmatrix}, \begin{pmatrix} a^2\theta + \beta\boldsymbol{I} & a\theta\mathbf{1} \\ a\theta\mathbf{1}^t & \theta \end{pmatrix}^{-1} \right)$$

$$\begin{array}{ll}
y|\beta, x_0, \theta & \sim \mathbf{N}\left(ax_0, \frac{\theta\beta}{\theta + a^2\beta}\right) \\
\boldsymbol{y}|\beta, x_0, \theta & \sim \mathbf{N}(ax_0\mathbf{1}, \frac{\theta\beta}{\theta + a^2\beta}\boldsymbol{I})
\end{array}$$

$$x|\boldsymbol{y}, \beta, x_0, \theta \sim \mathbf{N}(x_m, \theta_m),$$

avec

$$x_m = \frac{1}{\theta_m}(\theta x_0 + ma\beta\bar{y}) \longrightarrow x_m = \frac{1}{\theta_m}(\theta_{m-1}x_0 + a\beta y_m)$$

$$\theta_m = (\theta + ma^2\beta) \longrightarrow \quad \theta_m = \theta_{m-1} + a^2\beta$$

$$\bar{y} = \frac{1}{m}\sum_i y_i$$

$$\begin{array}{ll}
y_k|\boldsymbol{y}, \beta, x_0, \theta & \sim \mathbf{N}\left(ax_m, \frac{\theta_m\beta}{\theta_m + ma^2\beta}\right) \\
b_k|\boldsymbol{y}, \beta, x_0, \theta & \sim \mathbf{N}\left(0, \frac{2\theta_m\beta}{2\theta_m + ma^2\beta}\right)
\end{array}$$

$$\begin{array}{llll}
\beta = 0 & \longrightarrow & x_m = x_0, & \theta_m = \theta \\
\theta = 0 & \longrightarrow & x_m = \bar{y}/a, & \theta_m = ma^2\beta \\
\beta \longrightarrow \infty & \longrightarrow & x_m = \bar{y}/a, & \theta_m = \infty \\
\theta \longrightarrow \infty & \longrightarrow & x_m = x_0, & \theta_m = \infty \\
m \longrightarrow \infty & \longrightarrow & x_m = \bar{y}/a, & \theta_m = \infty
\end{array}$$

$$\begin{array}{llll}
\theta = r\beta & \longrightarrow & x_m = \frac{1}{r + ma^2}(ma\bar{y} + rx_0), & \theta_m = (r + ma^2)\beta \\
r \longrightarrow \infty & \longrightarrow & x_m = x_0, & \theta_m = \infty \\
r = 0 & \longrightarrow & x_m = \bar{y}/a, & \theta_m = ma^2\beta
\end{array}$$

Problème 5 :

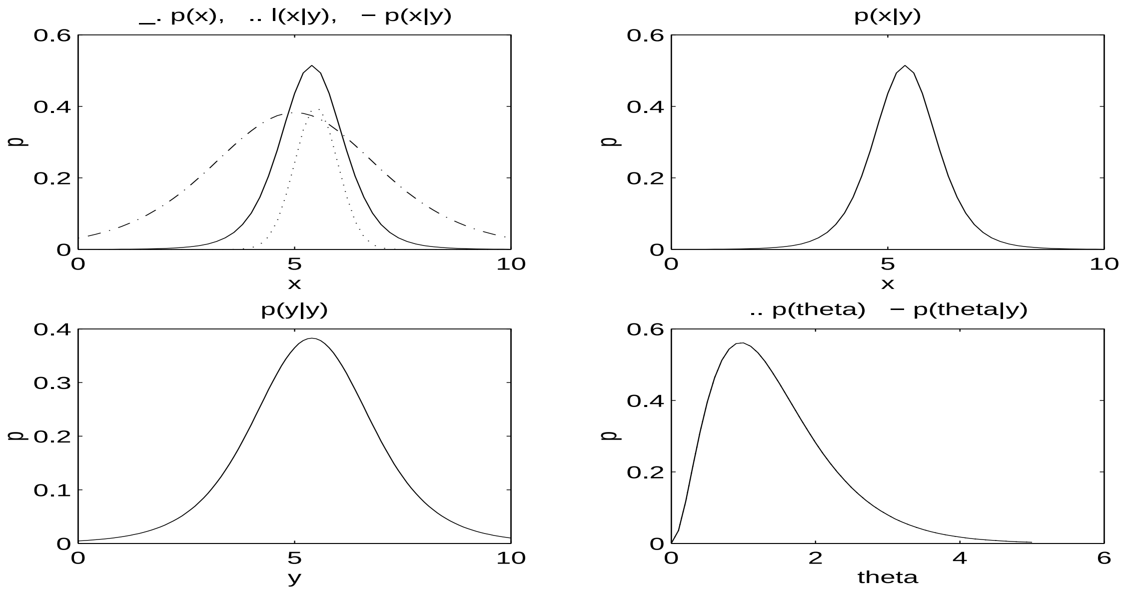
$$
\begin{array}{ll}
y_i = x + b_i, & i = 1, \cdots, m \quad \longrightarrow \boldsymbol{y} = ax\mathbf{1} + \boldsymbol{b} \\
\\
b_i|\beta & \sim \mathbf{N}(0, \beta) \\
y_i|x, \beta & \sim \mathbf{N}(x, \beta) \\
\\
x|x_0, \theta & \sim \mathbf{N}(x_0, \theta), \qquad \theta = r\beta, \quad r = \theta/\beta \\
\theta|a, b & \sim \mathbf{Gam}(a, rb), \quad a = r_0 b, \quad r_0 = a/b, \quad \mathrm{E}[\theta] = \frac{a}{rb} = \frac{r_0}{r}, \quad \mathrm{Var}[\theta] = \frac{a}{(rb)^2} \\
(x, \theta) & \sim \mathbf{NGam}(x_0, 1, a, rb) \\
\\
x|x_0, r, \beta & \sim \mathbf{N}(x_0, r\beta), \\
\beta|a, b & \sim \mathbf{Gam}(a, b), \quad a = r_0 b, \quad r_0 = a/b, \quad \mathrm{E}[\beta] = \frac{a}{b} = r_0, \quad \mathrm{Var}[\beta] = \frac{a}{b^2} = \frac{r_0}{b} \\
(x, \beta) & \sim \mathbf{NGam}(x_0, r, a, b) \\
\\
x|x_0, r, r_0, a & \sim \mathbf{St}\left(x_0, r r_0, 2a\right), \quad \mathrm{E}[x] = x_0, \quad \mathrm{Var}[x] = \frac{2a}{2a-2}\frac{1}{r r_0} \\
y_i|x_0, r, r_0, a & \sim \mathbf{St}\left(x_0, \frac{r}{r+1} r_0, 2a\right), \quad \mathrm{E}[y] = x_0, \quad \mathrm{Var}[y] = \frac{2a}{2a-2}\frac{r+1}{r r_0} \\
\\
x|\boldsymbol{y}, x_0, r, r_0, a & \sim \mathbf{St}\left(x_m, (m+r) r_m, 2a_m\right), \\
& \text{avec} \\
& x_m = \frac{1}{r+m}(r x_0 + m\bar{y}) \longrightarrow x_m = \frac{1}{r+m}[(r+m-1)x_{m-1} + y_m] \\
& r_m = a_m / b_m \\
& a_m = a + m/2, \quad b_m = b + m s^2/2 + \frac{1}{2}\frac{rm}{r+m}(x_0 - \bar{y})^2 \\
& \bar{y} = \frac{1}{m}\sum_{i=1}^m y_i, \quad s^2 = \sum_{i=1}^m (y_i - \bar{y})^2 \\
\\
& \mathrm{E}[x|\boldsymbol{y}] = x_m, \text{ si } 2a_m > 1 \\
& \mathrm{Var}[x|\boldsymbol{y}] = \frac{1}{(m+r)r_m}\frac{2a_m}{2a_m - 2}, \text{ si } 2a_m > 2 \\
& \mathrm{mode}(x|\boldsymbol{y}) = x_m \\
\\
y_k|\boldsymbol{y}, x_0, r, r_0, a & \sim \mathbf{St}\left(x_m, \frac{m+r}{m+r+1} r_m, 2a_m\right) \\
\beta|\boldsymbol{y}, x_0, r, r_0, a & \sim \mathbf{Gam}(a_m, b_m) \\
\theta|\boldsymbol{y}, x_0, r, r_0, a & \sim \mathbf{Gam}(a_m, r b_m) \\
& \mathrm{E}[\beta|\boldsymbol{y}] = \frac{a_m}{b_m}, \quad \mathrm{Var}[\beta|\boldsymbol{y}] = \frac{a_m}{b_m^2}, \quad \mathrm{mode}(\beta|\boldsymbol{y}) = \frac{a_m - 1}{b_m} \\
& \mathrm{E}[\theta|\boldsymbol{y}] = \frac{a_m}{r b_m}, \quad \mathrm{Var}[\theta|\boldsymbol{y}] = \frac{a_m}{r^2 b_m^2}, \quad \mathrm{mode}(\theta|\boldsymbol{y}) = \frac{a_m - 1}{r b_m}
\end{array}
$$

$$
\begin{array}{lll}
\theta = 0 \longrightarrow r = 0 & \longrightarrow & x_m = \bar{y}, \quad b_m = b + m s^2/2 \\
\theta \longrightarrow \infty & \longrightarrow & x_m = x_0, \quad b_m = b + m s^2/2 + \frac{m}{2}(x_0 - \bar{y})^2 = b + \frac{m}{2}[s^2 + (x_0 - \bar{y})^2] \\
m \longrightarrow \infty & \longrightarrow & x_m = \bar{y}, \quad x|\boldsymbol{y}, x_0, r, r_0, a \sim \mathbf{N}(\bar{y}, 0)
\end{array}
$$

Cas particulier :

$$
\begin{array}{llll}
x_0 = \bar{y} & \longrightarrow & x_m = \bar{y} & a_m = a + m/2, \quad b_m = b + (m/2)s^2, \\
x_0 = \bar{y}, a = \frac{1}{2}, b = \frac{s^2}{2} & \longrightarrow & x_m = \bar{y}, & a_m = \frac{m+1}{2}, b_m = \frac{m+1}{2}s^2, \\
& & r_m = \frac{1}{s^2}, & \\
& & \mathrm{Var}[x|\boldsymbol{y}] = \frac{m+1}{m-1}\frac{1}{m+r}s^2 &
\end{array}
$$

Exemple 5.1 : $\boldsymbol{y} = [6, 5], \quad m = 2, \quad \bar{y} = 5.5, \quad s^2 = 0.25,$
$\beta = 1, x_0 = 5, \boldsymbol{\theta} = .5, r = .5, \quad a = 2, b = 4, r_0 = .5, \quad \lambda_0 = 0.25, \mathrm{Var}\,[x] = 6$
$a_m = 3, b_m = 4.15, r_m = 0.7229, \quad x_m = 5.4, \lambda_m = 1.8072, \mathrm{Var}\,[x|\boldsymbol{y}] = .83$



Exemple 5.2 : $\boldsymbol{y} = [6, 5, 4, 7, 3, 5, 6, 8, 7, 6], \quad m = 10, \quad \bar{y} = 5.7, \quad s^2 = 2.01,$
$\beta = 1, x_0 = 5, \boldsymbol{\theta} = .5, r = .5, \quad a = 2, b = 4, r_0 = .5, \quad \lambda_0 = 0.25, \mathrm{Var}\,[x] = 4.667$
$a_m = 7, b_m = 13.88, r_m = 0.5042, \quad x_m = 5.667, \lambda_m = 5.294, \mathrm{Var}\,[x|\boldsymbol{y}] = .2204$