Annexe F

Quelques transformations utiles

L'objet de cette annexe est de rappeler un certain nombre de transformation linéaires utiles en 1-D, 2-D et n-D. Nous donnons aussi l'expression de la transformée inverse de ces transformations lorsqu'elle existe.

F.1 Transformations 1-D

$$F(u) = \int f(x) h(u,x)\, \mathrm{d}x$$

Identité : $h(u,x) = \delta(u-x)$ $\quad F(u) = \int f(x)\delta(u-x)\, \mathrm{d}x = f(x)$

Convolution : $h(u,x) = h(u-x)$ $\quad F(u) = \int f(x) h(u-x)\, \mathrm{d}x = f(x) * h(x)$

Corrélation : $h(u,x) = h(u+x)$ $\quad F(u) = \int f(x) h(u+x)\, \mathrm{d}x = f(x) \star h(x)$

Fourier : $h(u,x) = \exp\left[-jux\right]$ $\quad F(u) = \int f(x) \exp\left[-jux\right] \mathrm{d}x$

Laplace : $h(u,x) = \exp\left[-ux\right]$ $\quad F(u) = \int f(x) \exp\left[-ux\right] \mathrm{d}x$

Transformations unitaires :

$$F(u) = \int f(x) h(u,x)\,\mathrm{d}x$$
$$f(x) = \int F(u) h^*(u,x)\,\mathrm{d}u$$

Fourier : $h(u,x) = \exp[-jux]$

$$F(u) = \frac{1}{\sqrt{2\pi}} \int f(x) \exp[-jux]\,\mathrm{d}x$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int F(u) \exp[+jux]\,\mathrm{d}u$$

Cosinus : $h(u,x) = \cos(ux)$

$$F(u) = \frac{2}{\sqrt{\pi}} \int_0^\infty f(x) \cos(ux)\,\mathrm{d}x$$
$$f(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty F(u) \cos(ux)\,\mathrm{d}u$$

Sinus : $h(u,x) = \sin(ux)$

$$F(u) = \frac{2}{\sqrt{\pi}} \int_0^\infty f(x) \sin(ux)\,\mathrm{d}x$$
$$f(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty F(u) \sin(ux)\,\mathrm{d}u$$

Hartley : $h(u,x) = \frac{1}{\sqrt{2\pi}}[\cos(ux) + \sin(ux)]$

$$F(u) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x)[\cos(ux) + \sin(ux)]\,\mathrm{d}x$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty F(u)[\cos(ux) + \sin(ux)]\,\mathrm{d}u$$

Hilbert : $h(u,x) = \frac{1}{\pi}(u-x)^{-1}$

$$F(u) = \frac{1}{\pi} \int \frac{f(x)}{(x-u)}\,\mathrm{d}x$$
$$f(x) = \frac{1}{\pi} \int \frac{F(u)}{(x-u)}\,\mathrm{d}u$$

Hankel : $h(u,x) = J_\nu(ux)$

$$F(u) = \int_0^\infty f(x) J_\nu(ux)\,\mathrm{d}x$$
$$f(x) = \int_0^\infty F(u) J_\nu(ux)\,\mathrm{d}u$$

Mellin : $h(u,x) = x^{u-1}$

$$F(u) = \int f(x) x^{u-1}\,\mathrm{d}x$$

Abel : $h(u,x) = \frac{1}{\Gamma(\alpha)}(x-u)^{\alpha-1}$

$$F(u) = \frac{1}{\Gamma(\alpha)} \int f(x)(x-u)^{\alpha-1}\,\mathrm{d}x$$

Weyl : $h(u,x) = \frac{1}{\Gamma(\alpha)}(u-x)^{\alpha-1}$

$$F(u) = \frac{1}{\Gamma(\alpha)} \int f(x)(u-x)^{\alpha-1}\,\mathrm{d}x$$

F.2 Transformations 2-D

$$F(u,v) = \iint f(x,y)h(u,v;x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Identité :
$$h(u,v;x,y) = \delta(u-x;v-y)$$
$$F(u,v) = \iint f(x,y)\delta(u-x;v-y)\,\mathrm{d}x\,\mathrm{d}y = f(x,y)$$

Séparable :
$$h(u,v;x,y) = h_1(u,x)h_2(v,y)$$
$$F(u,v) = \iint f(x,y)h_1(u,x)h_2(v,y)\,\mathrm{d}x\,\mathrm{d}y$$

Séparable & Symétrique :
$$h(u,v;x,y) = h_1(u,x)h_1(v,y)$$
$$F(u,v) = \iint f(x,y)h_1(u,x)h_1(v,y)\,\mathrm{d}x\,\mathrm{d}y$$

Convolution :
$$h(u,v;x,y) = h(u-x;v-y)$$
$$F(u,v) = \iint f(x,y)h(u-x;v-y)\,\mathrm{d}x\,\mathrm{d}y = f(x,y) * h(x,y)$$

Corrélation :
$$h(u,v;x,y) = h(u+x;v+y)$$
$$F(u,v) = \iint f(x,y)h(u+x;v+y)\,\mathrm{d}x\,\mathrm{d}y = f(x,y) \star h(x,y)$$

Laplace :
$$h(u,v;x,y) = \exp\left[-(ux+vy)\right]$$
$$F(u,v) = \iint f(x,y)\exp\left[-(ux+vy)\right]\,\mathrm{d}x\,\mathrm{d}y$$

Transformations 2-D unitaires

Fourier : $h(u,v;x,y) = \exp\left[-j(ux+vy)\right]$

$$F(u,v) = \iint f(x,y)\exp\left[-j(ux+vy)\right]\,\mathrm{d}x\,\mathrm{d}y$$

Cosinus : $h(u,v;x,y) = \cos(ux)\cos(vy)$

$$F(u,v) = \iint f(x,y)\cos(ux)\cos(vy)\,\mathrm{d}x\,\mathrm{d}y$$

Sinus : $h(u,v;x,y) = \sin(ux)\sin(vy)$

$$F(u,v) = \iint f(x,y)\sin(ux)\sin(vy)\,\mathrm{d}x\,\mathrm{d}y$$

Abel : $h(u,v;x,y) = [(x-u)^2+(y-v)^2]^{(\alpha-1)/2}$

$$F(u,v) = \iint f(x,y)[(u-x)^2+(v-y)^2]^{-(\alpha-1)/2}\,\mathrm{d}x\,\mathrm{d}y$$

Weyl : $h(u,v;x,y) = [(u-x)^2+(v-y)^2]^{-(\alpha-1)/2}$

$$F(u,v) = \iint f(x,y)[(u-x)^2+(v-y)^2]^{-(\alpha-1)/2}\,\mathrm{d}x\,\mathrm{d}y$$

Hilbert : $h(u,v;x,y) = [(u-x)^2+(v-y)^2]^{-1/2}$

$$F(u,v) = \iint \frac{f(x,y)}{\sqrt{(u-x)^2+(v-y)^2}}\,\mathrm{d}x\,\mathrm{d}y$$

Radon : $h(r,\phi;x,y) = \delta(r - x\cos\phi - y\sin\phi)$

$$F(r,\phi) = \iint f(x,y)\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

F.3 Transformations n-D

$$F(\boldsymbol{u}) = \int f(\boldsymbol{x}) h(\boldsymbol{u}, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

Identité : $h(\boldsymbol{u}, \boldsymbol{x}) = \delta(\boldsymbol{u} - \boldsymbol{x})$ $\quad F(\boldsymbol{u}) = \int f(\boldsymbol{x}) \delta(\boldsymbol{u} - \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = f(\boldsymbol{x})$

Convolution : $h(\boldsymbol{u}, \boldsymbol{x}) = h(\boldsymbol{u} - \boldsymbol{x})$ $\quad F(\boldsymbol{u}) = \int f(\boldsymbol{x}) h(\boldsymbol{u} - \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = f(\boldsymbol{x}) * h(\boldsymbol{x})$

Corrélation : $h(\boldsymbol{u}, \boldsymbol{x}) = h(\boldsymbol{u} + \boldsymbol{x})$ $\quad F(\boldsymbol{u}) = \int f(\boldsymbol{x}) h(\boldsymbol{u} + \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = f(\boldsymbol{x}) \star h(\boldsymbol{x})$

Laplace : $h(\boldsymbol{u}, \boldsymbol{x}) = \exp\left[-\boldsymbol{u}^t \boldsymbol{x}\right]$ $\quad F(\boldsymbol{u}) = \int f(\boldsymbol{x}) \exp\left[-\boldsymbol{u}^t \boldsymbol{x}\right] \, \mathrm{d}\boldsymbol{x}$

Transformations unitaires

$$F(\boldsymbol{u}) = \int f(\boldsymbol{x}) h(\boldsymbol{u}, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$
$$f(\boldsymbol{x}) = \int F(\boldsymbol{u}) h^*(\boldsymbol{u}, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{u}$$

Fourier : $h(\boldsymbol{u}, \boldsymbol{x}) = \exp\left[-j \boldsymbol{u}^t \boldsymbol{x}\right]$ $\quad F(\boldsymbol{u}) = \int f(\boldsymbol{x}) \exp\left[-j \boldsymbol{u}^t \boldsymbol{x}\right] \, \mathrm{d}\boldsymbol{x}$

$$f(\boldsymbol{x}) = \int F(\boldsymbol{u}) \exp\left[+j \boldsymbol{u}^t \boldsymbol{x}\right] \, \mathrm{d}\boldsymbol{u}$$

Abel : $h(\boldsymbol{u}, \boldsymbol{x}) = ||\boldsymbol{x} - \boldsymbol{u}||^{(\alpha-1)/2}$ $\quad F(\boldsymbol{u}) = \int f(\boldsymbol{x}) ||\boldsymbol{x} - \boldsymbol{u}||^{(\alpha-1)/2} \, \mathrm{d}\boldsymbol{x}$

Weyl : $h(\boldsymbol{u}, \boldsymbol{x}) = ||\boldsymbol{u} - \boldsymbol{x}||^{(\alpha-1)/2}$ $\quad F(\boldsymbol{u}) = \int f(\boldsymbol{x}) ||\boldsymbol{u} - \boldsymbol{x}||^{(\alpha-1)/2} \, \mathrm{d}\boldsymbol{x}$

Hilbert : $h(\boldsymbol{u}, \boldsymbol{x}) = ||\boldsymbol{u} - \boldsymbol{x}||^{-1/2}$ $\quad F(\boldsymbol{u}) = \int f(\boldsymbol{x}) ||\boldsymbol{u} - \boldsymbol{x}||^{-1/2} \, \mathrm{d}\boldsymbol{x}$

Radon : $h(r, \boldsymbol{\xi}; \boldsymbol{x}) = \delta(r - \boldsymbol{\xi}^t \boldsymbol{x})$ $\quad F(r, \boldsymbol{\xi}) = \int f(\boldsymbol{x}) \delta(r - \boldsymbol{\xi}^t \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$

F.4 Transformations linéaires dans le cas discret

$$g(m)=\sum_{n=0}^{N-1} f(n)h(m,n), \quad m=0,\cdots,N-1 \quad \longrightarrow \quad \boldsymbol{g}=\boldsymbol{H}\boldsymbol{f}$$

Identité : $h(m,n)=\delta(m-n), \quad m,n=0,\cdots,N-1$

$\boldsymbol{H}$ matrice Identité

$$\boldsymbol{H}=\begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & 1 & & & & \vdots \\ \vdots & & & & & \\ & & & & & \\ & & & & & \vdots \\ \vdots & & & & & 0 \\ 0 & \cdots & & \vdots & 0 & 1 \end{pmatrix}$$

Convolution : $h(m,n)=h(m-n), \quad m,n=0,\cdots,N-1$

$\boldsymbol{H}$ matrice de Toeplitz

$$\boldsymbol{H}=\begin{pmatrix} h_0 & h_{-1} & h_{-2} & \cdots & & \cdots & h_{-N+1} \\ h_1 & h_0 & h_{-1} & & & & \vdots \\ h_2 & h_1 & & & & & \vdots \\ \vdots & & & & & & \\ \vdots & & & & & & h_{-2} \\ \vdots & & & & & & h_{-1} \\ h_{N-1} & \cdots & & & h_2 & h_1 & h_0 \end{pmatrix}$$

Corrélation : $h(m,n)=h(m+n), \quad m,n=0,\cdots,N-1$

$\boldsymbol{H}$ matrice de Hankel

$$\begin{pmatrix} h_0 & h_1 & h_2 & & h_3 & \cdots & h_{N-1} \\ h_1 & h_2 & h_3 & & & h_{N-1} & \vdots \\ h_2 & h_3 & & & & & \\ h_3 & & & & & & \\ \vdots & & & & & & \vdots \\ & & & & & & h_{2N-4} \\ \vdots & h_{N-1} & & & & h_{2N-4} & h_{2N-3} \\ h_{N-1} & \cdots & & & h_{2N-4} & h_{2N-3} & h_{2N-2} \end{pmatrix}$$

Transformations unitaires dans le cas discret

Fourier : $h(m,n) = \frac{1}{N}\exp\left[-j2\pi mn/N\right], \quad m,n = 0,\cdots,N-1$

$\boldsymbol{H}$ matrice circulante de TFD

$$\boldsymbol{H} = \frac{1}{N}\begin{pmatrix} 1 & 1 & 1 & \cdots & & \cdots & 1 \\ 1 & w^1 & w^2 & & & & w^{N-1} \\ 1 & w^2 & w^4 & & & & w^{2(N-1)} \\ \vdots & & & & & & \\ & & & & & & w^{(N-3)(N-1)} \\ \vdots & & & & & & w^{(N-2)(N-1)} \\ 1 & w^{N-1} & & & & w^{(N-2)(N-1)} & w^{(N-1)(N-1)} \end{pmatrix}$$

avec $w = \exp\left[-j2\pi/N\right]$

Cosinus : $h(m,n) = \begin{cases} \frac{1}{\sqrt{N}} & \text{si } m = 0 \\ \frac{2}{\sqrt{N}}\cos(\pi m(n+1/2)/N) & \text{si } m \neq 0 \end{cases}, \quad m,n = 0,\cdots,N-1$

$\boldsymbol{H}$ matrice circulante de TCD

$$\boldsymbol{H} = \begin{pmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & \cos(w) & \cos(2w) & & & \cos[(N-1)w] \\ 1 & \cos(2w) & & & & \vdots \\ \vdots & & & & & \\ & & & & & \vdots \\ & & & & & \cos[(N-3)(N-1)w] \\ \vdots & & & & & \cos[(N-2)(N-1)w] \\ 1 & \cos[(N-1)w] & & & & \cos[(N-1)(N-1)w] \end{pmatrix}$$

avec $w = -j2\pi/N$

Sinus : $h(m,n) = \frac{2}{\sqrt{N}}\sin(\pi m(n+1/2)/N), \quad m,n = 0,\cdots,N-1$

$\boldsymbol{H}$ matrice circulante de TSD

$$\boldsymbol{H} = \begin{pmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & \sin(w) & \sin(2w) & & & \sin[(N-1)w] \\ 1 & \sin(2w) & & & & \vdots \\ \vdots & & & & & \\ & & & & & \vdots \\ & & & & & \sin[(N-3)(N-1)w] \\ \vdots & & & & & \sin[(N-2)(N-1)w] \\ 1 & \sin[(N-1)w] & & & & \sin[(N-1)(N-1)w] \end{pmatrix}$$

avec $w = -j2\pi/N$

Walsh : $h(m,n) = \prod_{i=0}^{k} (-1)^{b_i(m) b_{n-1-i}(n)}, \quad N = 2^k$
$\boldsymbol{H}$ matrice symétrique avec des éléments -1 ou +1
$b_k(z)$ est le k ème bit de la représentation binaire de z.

$$\boldsymbol{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

Hadamard : $h(m,n) = (-1)^{\sum_{i=0}^{k} b_i(m) b_i(n)}, \quad N = 2^k$
$\boldsymbol{H}$ matrice symétrique avec des éléments -1 ou +1
$b_k(z)$ est le k ème bit de la représentation binaire de z.
Exemple: $z = 6 = 110 \longrightarrow b_0(z) = 0, b_1(z) = 1, b_2(z) = 2$.

$$\boldsymbol{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$