Gauss-Markov-Potts Priors for Inverse Problems in Imaging Systems

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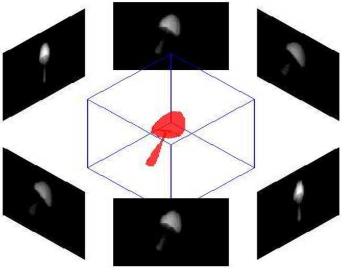
Content

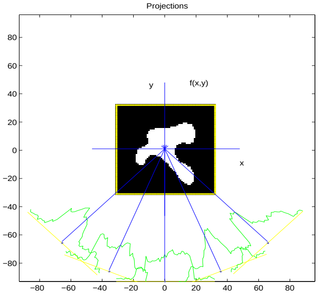
- Computed Tomography (CT) as an Invers Problem example
- Classical methods : analytical and algebraic method
- Probabilistic methods
- Bayesian inference approach
- Gauss-Markov-Potts prior moedels for images
- Bayesian computation
- VB with Gauss-Markov-Potts prior moedels
- Application in Computed Tomography
- Conclusions
- Questions and Discussion

2D and 3D Computed Tomography

3D



2D



$$g_\phi(r_1, r_2) = \int_{\mathcal{L}_{r_1,r_2,\phi}} f(x,y,z) \, \mathrm{d}l \qquad g_\phi(r) = \int_{\mathcal{L}_{r,\phi}} f(x,y) \, \mathrm{d}l$$

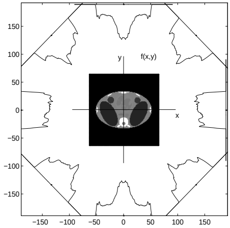
Forward probelm : $f(x,y)$ or $f(x,y,z) \longrightarrow g_\phi(r)$ or $g_\phi(r_1,r_2)$

Inverse problem : $g_\phi(r)$ or $g_\phi(r_1,r_2) \longrightarrow f(x,y)$ or $f(x,y,z)$

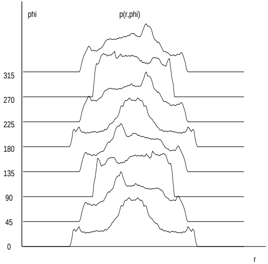
X ray Tomography and Radon Transform

$$g(r,\phi) = -\ln\left(\frac{I}{I_0}\right) = \int_{L_{r,\phi}} f(x,y)\, \mathrm{d}l$$

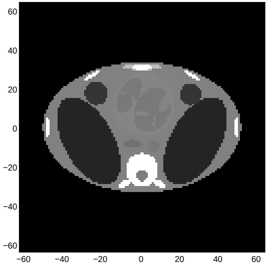
$$g(r,\phi) = \iint_D f(x,y)\, \delta(r - x\cos\phi - y\sin\phi)\, \mathrm{d}x\, \mathrm{d}y$$



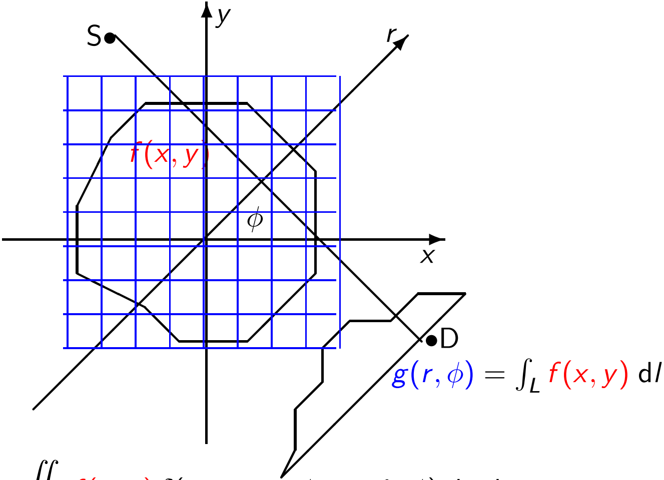
$f(x,y) \longrightarrow$ RT $\longrightarrow g(r,\phi)$



IRT
?
$\Longrightarrow$



Analytical Inversion methods



Radon :

$$g(r,\phi) = \iint_D f(x,y)\,\delta(r - x\cos\phi - y\sin\phi)\,\mathrm{d}x\,\mathrm{d}y$$

$$f(x,y) = \left(-\frac{1}{2\pi^2}\right)\int_0^{\pi}\int_{-\infty}^{+\infty}\frac{\frac{\partial}{\partial r}g(r,\phi)}{(r - x\cos\phi - y\sin\phi)}\,\mathrm{d}r\,\mathrm{d}\phi$$

Filtered Backprojection method

$$f(x,y) = \left(-\frac{1}{2\pi^2}\right) \int_0^{\pi} \int_{-\infty}^{+\infty} \frac{\frac{\partial}{\partial r} g(r,\phi)}{(r - x\cos\phi - y\sin\phi)} \, \mathrm{d}r \, \mathrm{d}\phi$$

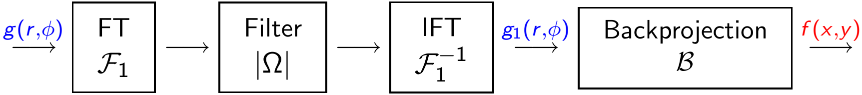
Derivation $\mathcal{D}$: $\bar{g}(r,\phi) = \dfrac{\partial g(r,\phi)}{\partial r}$

Hilbert Transform$\mathcal{H}$: $g_1(r',\phi) = \dfrac{1}{\pi} \displaystyle\int_0^{\infty} \frac{\bar{g}(r,\phi)}{(r - r')} \, \mathrm{d}r$

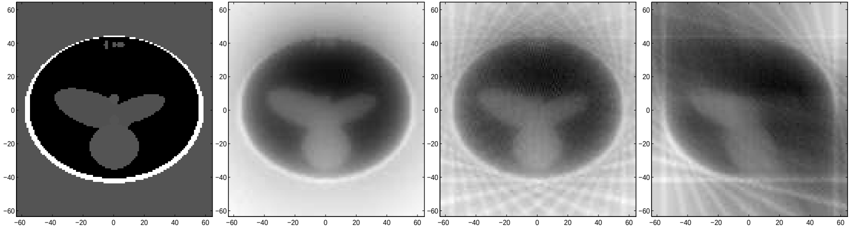
Backprojection $\mathcal{B}$: $f(x,y) = \dfrac{1}{2\pi} \displaystyle\int_0^{\pi} g_1(r' = x\cos\phi + y\sin\phi, \phi) \, \mathrm{d}\phi$

$$f(x,y) = \mathcal{B}\,\mathcal{H}\,\mathcal{D}\, g(r,\phi) = \mathcal{B}\,\mathcal{F}_1^{-1}\,|\Omega|\,\mathcal{F}_1\, g(r,\phi)$$

- Backprojection of filtered projections :



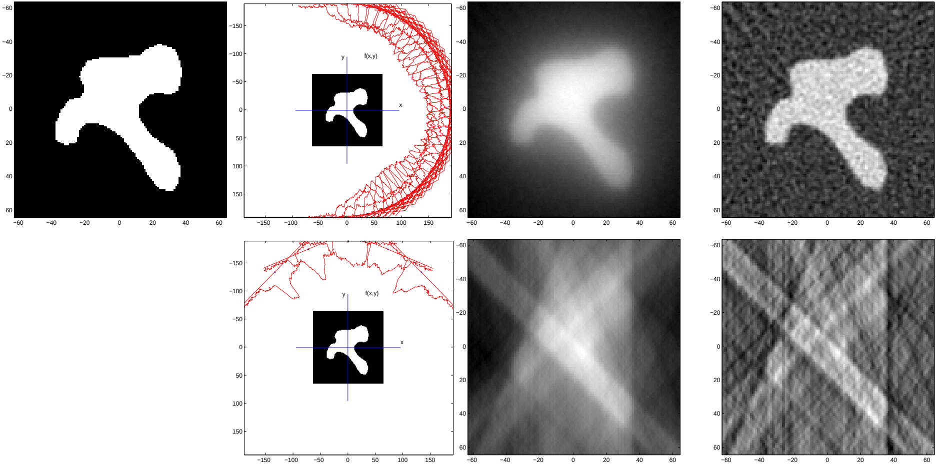
Limitations : Limited angle or noisy data



Original | 64 proj. | 16 proj. | 8 proj. $[0, \pi/2]$

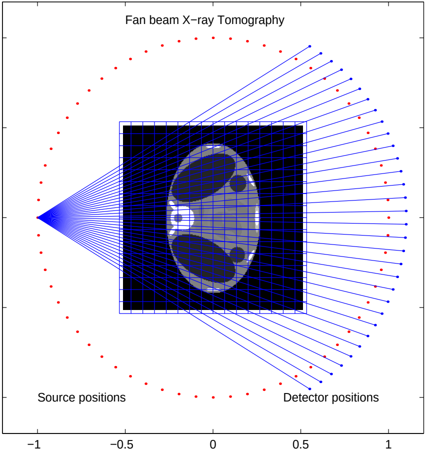
- Limited angle or noisy data
- Accounting for detector size
- Other measurement geometries : fan beam, ...

Limitations : Limited angle or noisy data



Original Data Backprojection Filtered Backprojectio

CT as a linear inverse problem



$$g(s_i) = \int_{L_i} f(r) \, \mathrm{d}l_i \longrightarrow \text{Discretization} \longrightarrow \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

Classical methods in CT

$$g(\boldsymbol{s}_i) = \int_{L_i} f(\boldsymbol{r}) \; \mathrm{d}l_i \longrightarrow \text{Discretization} \longrightarrow \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- $\boldsymbol{H}$ is a huge dimensional matrix of line integrals
- $\boldsymbol{H}\boldsymbol{f}$ is the forward or projection operation
- $\boldsymbol{H}^t\boldsymbol{g}$ is the backward or backprojection operation
- $(\boldsymbol{H}^t\boldsymbol{H})^{-1}\boldsymbol{H}^t\boldsymbol{g}$ is the filtered backprojection minimizing $\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2$
- Iterative methods :
  $\widehat{\boldsymbol{f}}^{(k+1)} = \widehat{\boldsymbol{f}}^{(k)} + \alpha^{(k)}\boldsymbol{H}^t\left(\boldsymbol{g} - \boldsymbol{H}\widehat{\boldsymbol{f}}^{(k)}\right)$
  is the Least squares iterative reconstruction method
- Regularization : $J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{D}\boldsymbol{f}\|^2$.

Inversion : Deterministic methods

Data matching

- Observation model
  $g_i = h_i(\boldsymbol{f}) + \epsilon_i, \quad i = 1, \ldots, M \longrightarrow \boldsymbol{g} = \boldsymbol{H}(\boldsymbol{f}) + \boldsymbol{\epsilon}$
- Misatch between data and output of the model $\Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f}))$

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{\Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f}))\}$$

- Examples :

– LS $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \|\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})\|^2 = \sum_i |g_i - h_i(\boldsymbol{f})|^2$

– L_p $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \|\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})\|^p = \sum_i |g_i - h_i(\boldsymbol{f})|^p, \quad 1 < p < 2$

– KL $\quad \Delta(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) = \sum_i g_i \ln \frac{g_i}{h_i(\boldsymbol{f})}$

- In general, does not give satisfactory results for inverse problems.

Regularization theory

Inverse problems = Ill posed problems
$\longrightarrow$ Need of prior information

- Minimum norme LS (MNLS) : $J(\boldsymbol{f}) = ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda ||\boldsymbol{f}||^2$
- Classical regularization : $J(\boldsymbol{f}) = ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda ||\boldsymbol{D}\boldsymbol{f}||^2$
$= ||\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})||^2 + \lambda \sum_j [\boldsymbol{D}\boldsymbol{f}]_j^2$
- More general regularization :

$$J(\boldsymbol{f}) = \mathcal{Q}(\boldsymbol{g} - \boldsymbol{H}(\boldsymbol{f})) + \lambda \sum_j \Phi([\boldsymbol{D}\boldsymbol{f}]_j)$$

or

$$J(\boldsymbol{f}) = \Delta_1(\boldsymbol{g}, \boldsymbol{H}(\boldsymbol{f})) + \lambda \Delta_2(\boldsymbol{f}, \boldsymbol{f}_\infty)$$

Limitations :

- Errors are considered implicitly white and Gaussian
- Limited prior information on the solution
- Lack of tools for the determination of the hyperparameters

Inversion : Probabilistic methods

Taking account of errors and uncertainties $\longrightarrow$ Probability theory

- Maximum Likelihood (ML)
- Minimum Inaccuracy (MI)
- Probability Distribution Matching (PDM)
- Maximum Entropy (ME) and Information Theory (IT)
- Bayesian Inference (Bayes)

Advantages :

- Explicit account of the errors and noise
- A large class of priors via explicit or implicit modeling
- A coherent approach to combine information content of the data and priors

Limitations :

- Practical implementation and cost of calculation

Bayesian estimation approach

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Observation model $\mathcal{M}$ + Hypothesis on the noise $\boldsymbol{\epsilon}$
  $\longrightarrow p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M}) = p_{\epsilon}(\boldsymbol{g}-\boldsymbol{H}\boldsymbol{f})$
- A priori information $\quad p(\boldsymbol{f}|\mathcal{M})$
- Bayes : $\quad p(\boldsymbol{f}|\boldsymbol{g};\mathcal{M}) = \dfrac{p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M})\,p(\boldsymbol{f}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$

Link with regularisation :

Maximum A Posteriori (MAP) :

$$\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{g}|\boldsymbol{f})\; p(\boldsymbol{f})\}$$
$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{-\ln p(\boldsymbol{g}|\boldsymbol{f}) - \ln p(\boldsymbol{f})\}$$

with

$$Q(\boldsymbol{g},\boldsymbol{H}\boldsymbol{f}) = -\ln p(\boldsymbol{g}|\boldsymbol{f}) \quad \text{and} \quad \lambda\Phi(\boldsymbol{f}) = -\ln p(\boldsymbol{f})$$

Case of linear models and Gaussian priors

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Hypothesis on the noise : $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma_\epsilon^2 \boldsymbol{I}) \longrightarrow$
  $p(\boldsymbol{g}|\boldsymbol{f}) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2\right\}$
- Hypothesis on $\boldsymbol{f}$: $\boldsymbol{f} \sim \mathcal{N}(0, \sigma_f^2 (\boldsymbol{D}^t\boldsymbol{D})^{-1}) \longrightarrow$
  $p(\boldsymbol{f}) \propto \exp\left\{-\frac{1}{2\sigma_f^2}\|\boldsymbol{D}\boldsymbol{f}\|^2\right\}$
- A posteriori :
  $$p(\boldsymbol{f}|\boldsymbol{g}) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 \frac{1}{2\sigma_f^2}\|\boldsymbol{D}\boldsymbol{f}\|^2\right\}$$
- MAP : $\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\}$
  with $J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{D}\boldsymbol{f}\|^2, \qquad \lambda = \frac{\sigma_\epsilon^2}{\sigma_f^2}$
- Advantage : caracterization of the solution

$$\boldsymbol{f}|\boldsymbol{g} \sim \mathcal{N}(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{P}}) \text{ with } \widehat{\boldsymbol{f}} = \widehat{\boldsymbol{P}}\boldsymbol{H}^t\boldsymbol{g}, \quad \widehat{\boldsymbol{P}} = \left(\boldsymbol{H}^t\boldsymbol{H} + \lambda\boldsymbol{D}^t\boldsymbol{D}\right)^{-1}$$

MAP estimation with other priors :

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \quad \text{avec} \quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\Omega(\boldsymbol{f})$$

Separable priors :

- Gaussian prior :

$$p(f_j) \propto \exp\left\{-\alpha(f_j - m_j)^2\right\} \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j (f_j - m_j)^2$$

- Gamma prior :

$$p(f_j) \propto (f_j/m_j)^\alpha \exp\left\{-f_j/m_j\right\} \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j \ln \frac{f_j}{m_j} + \frac{f_j}{m_j},$$

- Beta prior :

$$p(f_j) \propto f_j^\alpha (1 - f_j)^\beta \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha \sum_j \ln f_j + \beta \sum_j \ln(1 - f_j),$$

- Generalized gaussienne :

$$p(f_j) \propto \exp\left\{-\alpha|f_j - m_j|^p\right\}, \quad 1 < p < 2 \longrightarrow \quad \Phi(\boldsymbol{f}) = \alpha \sum_j |f_j - m_j|^p,$$

Markovian models :

$$p(f_j|\boldsymbol{f}_{-j}) \propto \exp\left\{-\alpha \sum_{i\in N_j} \phi(f_j, f_i)\right\} \longrightarrow \quad \Phi(\boldsymbol{f}) = \alpha \sum_j \sum_{i\in N_j} \phi(f_j, f_i)$$

MAP estimation with markovien priors :

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \quad \text{avec} \quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\Omega(\boldsymbol{f})$$

$$\Omega(\boldsymbol{f}) = \sum_j \phi(f_j - f_{j-1})$$

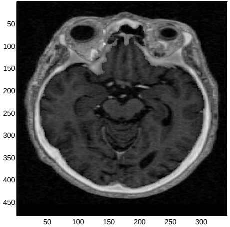
with $\phi(t)$:

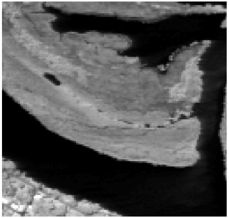
$$|t|^\alpha, \; \sqrt{1+t^2} - 1, \; \log(\cosh(t)), \; \left\{ \begin{array}{ll} t^2 & |t| \leq T \\ 2T|t| - T^2 & |t| > T \end{array} \right.$$

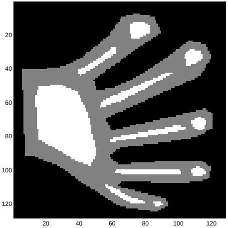
or

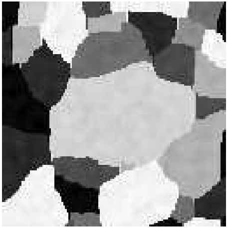
$$\log(1+t^2), \quad \frac{t^2}{1+t^2}, \quad \arctan(t^2), \quad \left\{ \begin{array}{ll} t^2 & |t| \leq T \\ T^2 & |t| > T \end{array} \right.$$

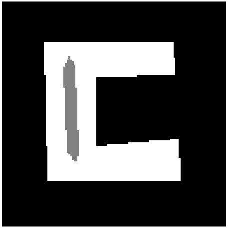
Which images I am looking for ?



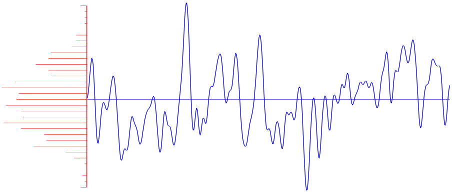




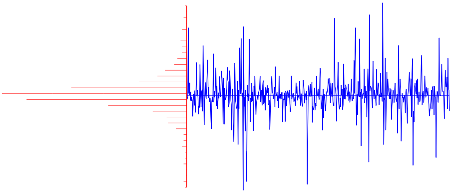




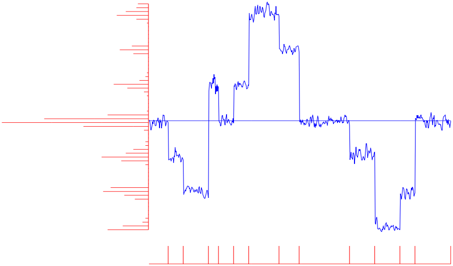
Which image I am looking for ?



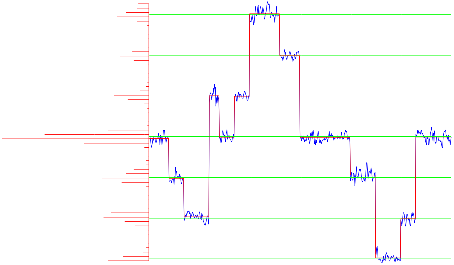
Gauss-Markov



Generalized GM



Piecewize Gaussian



Mixture of GM

Markovien prior models for images

$$\Omega(\boldsymbol{f}) = \sum_j \phi(f_j - f_{j-1})$$

- Gauss-Markov : $\phi(t) = |t|^2$
- Generalized Gauss-Markov : $\phi(t) = |t|^\alpha$
- Picewize Gauss-Markov or GGM : $\phi(t) = \begin{cases} t^2 & |t| \leq T \\ T^2 & |t| > T \end{cases}$

  or equivalently :

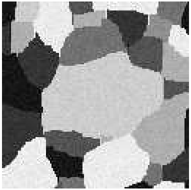
$$\Omega(\boldsymbol{f}|\boldsymbol{q}) = \sum_j (1 - q_j)\phi(f_j - f_{j-1})$$

  $\boldsymbol{q}$ line process (contours)
- Mixture of Gaussians :

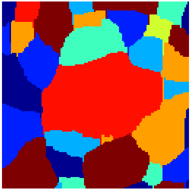
$$\Omega(\boldsymbol{f}|\boldsymbol{z}) = \sum_k \sum_{\{j:z_j=k\}} \left(\frac{f_j - m_k}{v_k}\right)^2$$

  $\boldsymbol{z}$ region labels process.

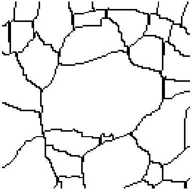
Gauss-Markov-Potts prior models for images



$f(\boldsymbol{r})$



$z(\boldsymbol{r})$



$q(\boldsymbol{r}) = 1 - \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))$

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}) = k, m_k, v_k) = \mathcal{N}(m_k, v_k)$$

$$p(f) = \sum_k P(z = k)\,\mathcal{N}(m_k, v_k) \quad \text{Mixture of Gaussians}$$

Separable iid hidden variables : $p(\boldsymbol{z}) = \prod_{\boldsymbol{r}} p(z(\boldsymbol{r}))$

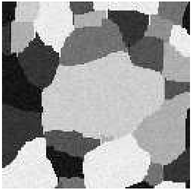
Markovian hidden variables : $p(\boldsymbol{z})$ Potts-Markov

$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

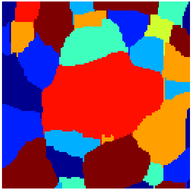
$$p(\boldsymbol{z}) \propto \exp\left\{\gamma \sum_{\boldsymbol{r} \in \mathcal{R}} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

Four different cases

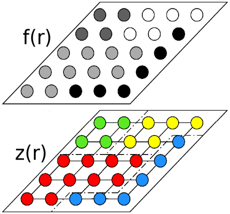
- $\boldsymbol{f}|\boldsymbol{z}$ iid, $\quad \boldsymbol{z}$ iid :
  Classical case of
  Mixture of Gaussians
- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\quad \boldsymbol{z}$ iid :
  (Markov composite)
  Mixture of Gauss-Markov
- $\boldsymbol{f}|\boldsymbol{z}$ iid, $\quad \boldsymbol{z}$ Markov :
  (Hidden Potts-Markov)
  Gauss-Potts
- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\quad \boldsymbol{z}$ Markov :
  (Gauss-Markov-Potts)

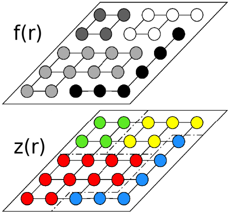


$f(\boldsymbol{r})$



$z(\boldsymbol{r})$





Case 1 : $\quad f|z$ iid, $\quad z$ iid

Independent Mixture of Independent Gaussiens (IMIG) :

$$
\begin{array}{l}
p(f(\boldsymbol{r})|z(\boldsymbol{r})=k)=\mathcal{N}(m_k,v_k), \forall \boldsymbol{r}\in\mathcal{R} \\
p(f(\boldsymbol{r}))=\sum_{k=1}^K \alpha_k \mathcal{N}(m_k,v_k), \text{with } \sum_k \alpha_k=1. \\
p(\boldsymbol{z})=\prod_{\boldsymbol{r}} p(z(\boldsymbol{r})=k)=\prod_{\boldsymbol{r}} \alpha_k=\prod_k \alpha_k^{n_k}
\end{array}
$$

Noting

$$
m_z(\boldsymbol{r})=m_k, v_z(\boldsymbol{r})=v_k, \alpha_z(\boldsymbol{r})=\alpha_k, \forall \boldsymbol{r}\in\mathcal{R}_k
$$

we have :

$$
\begin{aligned}
p(\boldsymbol{f}|\boldsymbol{z}) &= \prod_{\boldsymbol{r}\in\mathcal{R}} \mathcal{N}(m_z(\boldsymbol{r}),v_z(\boldsymbol{r})) \\
p(\boldsymbol{z}) &= \prod_{\boldsymbol{r}} \alpha_z(\boldsymbol{r}) = \prod_k \alpha_k^{\sum_{\boldsymbol{r}\in\mathcal{R}} \delta(z(\boldsymbol{r})-k)} = \prod_k \alpha_k^{n_k}
\end{aligned}
$$

Case 2 : $f|z$ Gauss-Markov, z iid

Independent Mixture of Gauss-Markov (IMGM) :

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}), z(\boldsymbol{r}'), f(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) = \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})), \forall \boldsymbol{r} \in \mathcal{R}$$

$$\begin{array}{ll} \mu_z(\boldsymbol{r}) & = \frac{1}{|\mathcal{V}(\boldsymbol{r})|} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \mu_z^*(\boldsymbol{r}') \\ \mu_z^*(\boldsymbol{r}') & = \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, f(\boldsymbol{r}') + (1 - \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, m_z(\boldsymbol{r}') \\ & = \quad (1 - c(\boldsymbol{r}')) \quad f(\boldsymbol{r}') \; + \qquad c(\boldsymbol{r}') \qquad m_z(\boldsymbol{r}') \end{array}$$

$$\begin{array}{lll} p(\boldsymbol{f}|\boldsymbol{z}) & \propto \prod_{\boldsymbol{r}} \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})) & \propto \prod_k \alpha_k \mathcal{N}(m_k 1, \boldsymbol{\Sigma}_k) \\ p(\boldsymbol{z}) & = \prod_{\boldsymbol{r}} v_z(\boldsymbol{r}) & = \prod_k \alpha_k^{n_k} \end{array}$$

with $1_k = 1, \forall \boldsymbol{r} \in \mathcal{R}_k$ and $\boldsymbol{\Sigma}_k$ a covariance matrix $(n_k \times n_k)$.

Case 3 : $f|z$ Gauss iid, z Potts

Gauss iid as in Case 1 :

$$p(\boldsymbol{f}|\boldsymbol{z}) = \prod_{\boldsymbol{r}\in\mathcal{R}} \mathcal{N}(m_z(\boldsymbol{r}), v_z(\boldsymbol{r})) = \prod_{k} \prod_{\boldsymbol{r}\in\mathcal{R}_k} \mathcal{N}(m_k, v_k)$$

Potts-Markov

$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

$$p(\boldsymbol{z}) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}\in\mathcal{R}} \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

Case 4 : $f|z$ Gauss-Markov, z Potts

Gauss-Markov as in Case 2 :

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}), z(\boldsymbol{r}'), f(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) = \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})), \forall \boldsymbol{r} \in \mathcal{R}$$

$$\begin{array}{ll} \mu_z(\boldsymbol{r}) & = \frac{1}{|\mathcal{V}(\boldsymbol{r})|} \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \mu_z^*(\boldsymbol{r}') \\ \mu_z^*(\boldsymbol{r}') & = \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, f(\boldsymbol{r}') + (1 - \delta(z(\boldsymbol{r}') - z(\boldsymbol{r}))\, m_z(\boldsymbol{r}') \end{array}$$

$$p(\boldsymbol{f}|\boldsymbol{z}) \quad \propto \prod_{\boldsymbol{r}} \mathcal{N}(\mu_z(\boldsymbol{r}), v_z(\boldsymbol{r})) \propto \prod_k \alpha_k \mathcal{N}(m_k 1, \boldsymbol{\Sigma}_k)$$

Potts-Markov as in Case 3 :

$$p(\boldsymbol{z}) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}\in\mathcal{R}} \sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

Full Bayesian approach

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Forward & errors model : $\longrightarrow p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})$
- Prior models $\longrightarrow p(\boldsymbol{f}|\boldsymbol{\theta};\mathcal{M})$ and $p(\boldsymbol{\theta}|\mathcal{M})$
- Bayes : $\longrightarrow p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g};\mathcal{M}) = \frac{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta};\mathcal{M})\, p(\boldsymbol{f}|\boldsymbol{\theta};\mathcal{M})\, p(\boldsymbol{\theta}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$
- Joint MAP :

$$(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{f},\boldsymbol{\theta})} \{p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g};\mathcal{M})\}$$

- Posterior means : $\begin{cases} \widehat{\boldsymbol{f}} & = \int \boldsymbol{f}\; p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g};\mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta} \\ \widehat{\boldsymbol{\theta}} & = \int \boldsymbol{\theta}\; p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g};\mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta} \end{cases}$
- Evidence of the model :

$$p(\boldsymbol{g}|\mathcal{M}) = \iint p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta};\mathcal{M}) p(\boldsymbol{f}|\boldsymbol{\theta};\mathcal{M}) p(\boldsymbol{\theta}|\mathcal{M})\, \mathrm{d}\boldsymbol{f}\, \mathrm{d}\boldsymbol{\theta}$$

Bayesian Computation

- Direct computation and use of $p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})$ is too complex
- Approximations :
  - Gauss-Laplace (Gaussian approximation)
  - Exploration (Sampling) using MCMC methods
  - Separable approximation (Variational techniques)
- Main idea :

  Approximate $p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}; \mathcal{M})$ by $q(\boldsymbol{f}, \boldsymbol{\theta}) = q_1(\boldsymbol{f})\ q_2(\boldsymbol{\theta})$
  - Choice of approximation criterion
  - Choice of appropriate families of probability laws for $q_1(\boldsymbol{f})$ and $q_2(\boldsymbol{\theta})$

Bayesian computation with Gauss-Markov-Potts prior models

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) = \frac{p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta})\; p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta})\; p(\boldsymbol{z})}{p(\boldsymbol{g}|\boldsymbol{\theta})}$$

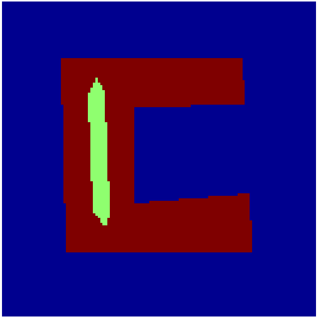
Approximations :

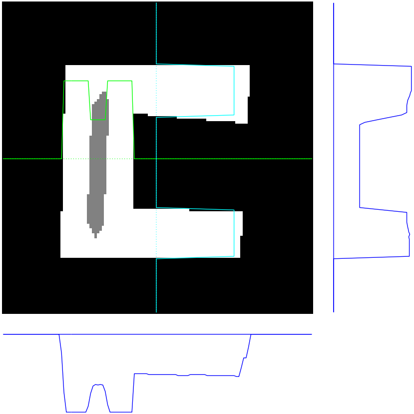
- $\boldsymbol{f}|\boldsymbol{z}$ iid, $\boldsymbol{z}$ iid :
  $q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) = q_1(\boldsymbol{f}|\boldsymbol{z})\; q_2(\boldsymbol{z})\; q_3(\boldsymbol{\theta})$.
- $\boldsymbol{f}|\boldsymbol{z}$ iid, $\boldsymbol{z}$ Markov :
  $q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) = q_1(\boldsymbol{f}|\boldsymbol{z})\; q_{2w}(\boldsymbol{z}_w)\; q_{2b}(\boldsymbol{z}_b)\; q_3(\boldsymbol{\theta})$.
- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\boldsymbol{z}$ iid :
  $q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) = q_{1w}(\boldsymbol{f}_w|\boldsymbol{z})\; q_{1b}(\boldsymbol{f}_b|\boldsymbol{z})\; q_2(\boldsymbol{z})\; q_3(\boldsymbol{\theta})$.
- $\boldsymbol{f}|\boldsymbol{z}$ Markov, $\boldsymbol{z}$ iid :

$$q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) = q_{1w}(\boldsymbol{f}_w|\boldsymbol{z})\; q_{1b}(\boldsymbol{f}_b|\boldsymbol{z})\; q_{2w}(\boldsymbol{z}_w)\; q_{2b}(\boldsymbol{z}_b)\; q_3(\boldsymbol{\theta})$$

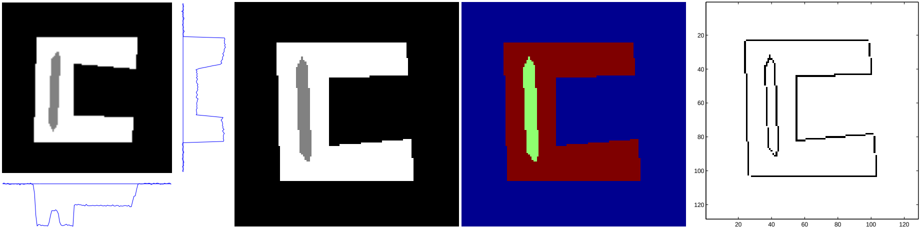
Application of CT in NDT

Reconstruction from only 2 projections





Application in CT



$\boldsymbol{g}\vert\boldsymbol{f}$	$\boldsymbol{f}\vert\boldsymbol{z}$	$\boldsymbol{z}$	$\boldsymbol{c}$
$\boldsymbol{g}=\boldsymbol{H}\boldsymbol{f}+\boldsymbol{\epsilon}$	iid Gaussian	iid	$c(\boldsymbol{r}) \in \{0,1\}$
$\boldsymbol{g}\vert\boldsymbol{f} \sim \mathcal{N}(\boldsymbol{H}\boldsymbol{f}, v_\epsilon \boldsymbol{I})$	or	or	$1-\delta(z(\boldsymbol{r})-z(\boldsymbol{r}'))$
Gaussian	Gauss-Markov	Potts	binary

Proposed algorithm

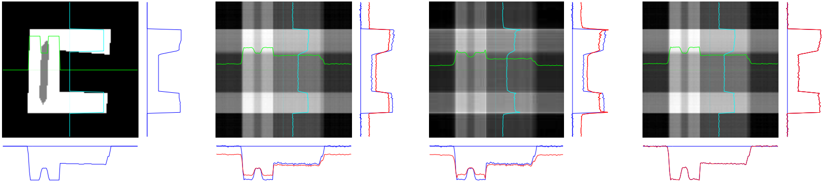
$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta})\, p(\boldsymbol{\theta})$$

General scheme :

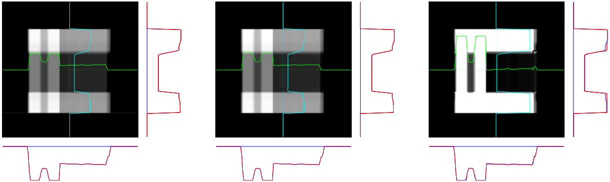
$$\widehat{\boldsymbol{f}} \sim p(\boldsymbol{f}|\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{z}} \sim p(\boldsymbol{z}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \longrightarrow \widehat{\boldsymbol{\theta}} \sim (\boldsymbol{\theta}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \boldsymbol{g})$$

- Estimate $\boldsymbol{f}$ using $p(\boldsymbol{f}|\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta})\, p(\boldsymbol{f}|\widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}})$
  Needs optimisation of a quadratic criterion.
- Estimate $\boldsymbol{z}$ using $p(\boldsymbol{z}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\theta}}, \boldsymbol{g}) \propto p(\boldsymbol{g}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\theta}})\, p(\boldsymbol{z})$
  Needs sampling of a Potts Markov field.
- Estimate $\boldsymbol{\theta}$ using
  $p(\boldsymbol{\theta}|\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{z}}, \boldsymbol{g}) \propto p(\boldsymbol{g}|\widehat{\boldsymbol{f}}, \sigma_\epsilon^2 \boldsymbol{I})\, p(\widehat{\boldsymbol{f}}|\widehat{\boldsymbol{z}}, (m_k, v_k))\, p(\boldsymbol{\theta})$
  Conjugate priors $\longrightarrow$ analytical expressions.

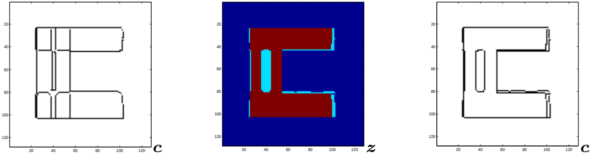
Results



Original Backprojection Filtered BP LS



Gauss-Markov+pos GM+Line process GM+Label process

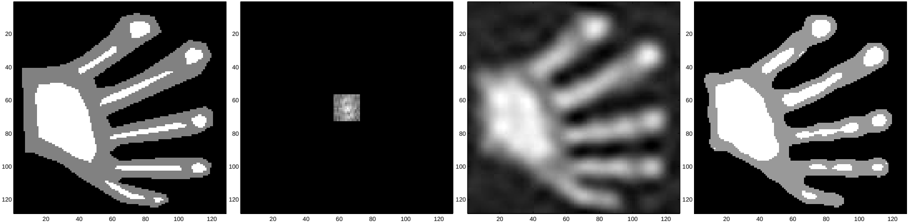


Application in Microwave imaging

$$g(\omega) = \int f(\boldsymbol{r}) \exp\{-j(\boldsymbol{\omega}.\boldsymbol{r})\} \, \mathrm{d}\boldsymbol{r} + \epsilon(\boldsymbol{\omega})$$

$$g(u,v) = \int f(x,y) \exp\{-j(ux+vy)\} \, \mathrm{d}x \, \mathrm{d}y + \epsilon(u,v)$$

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$



$f(x,y)$ $g(u,v)$ $\widehat{f}$ IFT $\widehat{f}$ Proposed method

Conclusions

- Bayesian Inference for inverse problems
- Approximations (Laplace, MCMC, Variational)
- Gauss-Markov-Potts are useful prior models for images incorporating regions and contours
- Separable approximations for Joint posterior with Gauss-Markov-Potts priors
- Application in different CT (X ray, US, Microwaves, PET, SPECT)

Perspectives :

- Efficient implementation in 2D and 3D cases
- Evaluation of performances and comparison with MCMC methods
- Application to other linear and non linear inverse problems : (PET, SPECT or ultrasound and microwave imaging)

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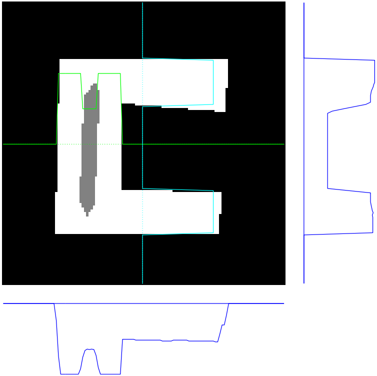
Questions and Discussions

- Thanks for your attentions
- ...
- ...
- Questions ?
- Discussions ?
- ...
- ...

CT from two projections = Joint distribution froms its marginals



$$g_1(x) = \int f(x,y)\,\mathrm{d}y$$

$$g_2(y) = \int f(x,y)\,\mathrm{d}x$$

Given the marginals $g_1(x)$ and $g_2(y)$ find the joint distribution $f(x,y)$

Infinite number of solutions

$$f(x,y) = g_1(x)\,g_2(y)\,\Omega(G_1(x), G_2(y))$$

$\Omega(u,v)$ is a Copula :

$\Omega(u,0) = 0,\ \Omega(u,1) = 1,$

$\Omega(0,u) = 0,\ \Omega(1,u) = 1$

$(x,y) \in [0,1]^2$, $G_1(x)$ and $G_2(y)$ are CDFs of $g_1(x)$ and $g_2(y)$

Example : $\Omega(u,v) = uv$

Any link between geometrical structure of f and Copula functions ?