

# RECONSTRUCTION D'IMAGE EN TOMOGRAPHIE X

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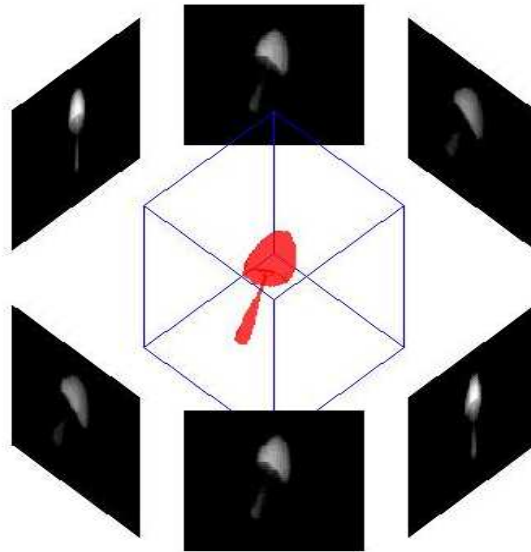
[djafari@lss.supelec.fr](mailto:djafari@lss.supelec.fr)

<http://djafari.free.fr>

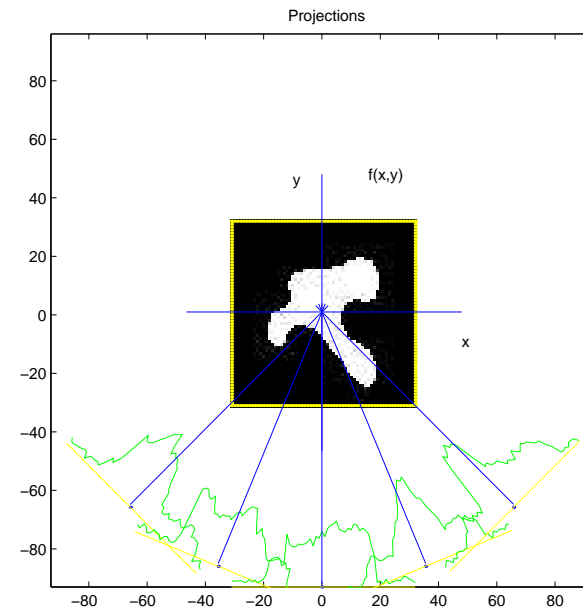
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# TOMOGRAPHIE X

## 3D



## 2D



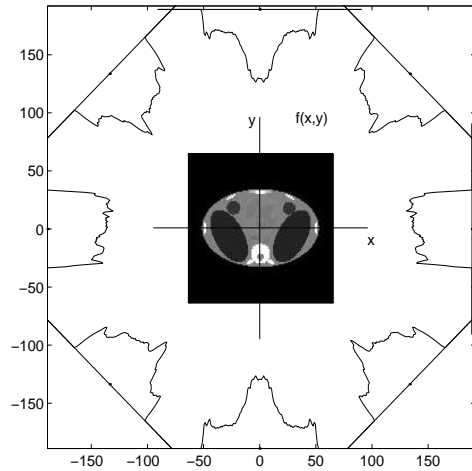
$$p_{\phi}(r_1, r_2) = \int_{\mathcal{L}_{r_1, r_2, \phi}} f(x, y, z) dl$$

$$p_{\phi}(r) = \int_{\mathcal{L}_{r, \phi}} f(x, y) dl$$

❑ Problème direct :  $f(x, y)$  ou  $f(x, y, z) \longrightarrow p_{\phi}(r)$  ou  $p_{\theta, \phi}(r_1, r_2)$

❑ Problème inverse :  $p_{\phi}(r)$  ou  $p_{\phi, \phi}(r_1, r_2) \longrightarrow f(x, y)$  ou  $f(x, y, z)$

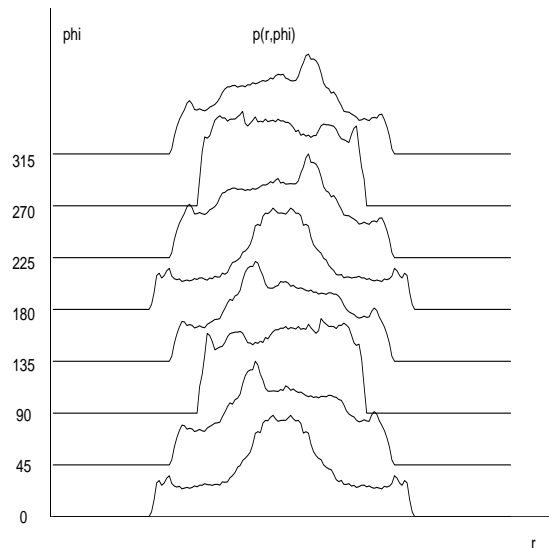
# RECONSTRUCTION D'IMAGE EN TOMOGRAPHIE X



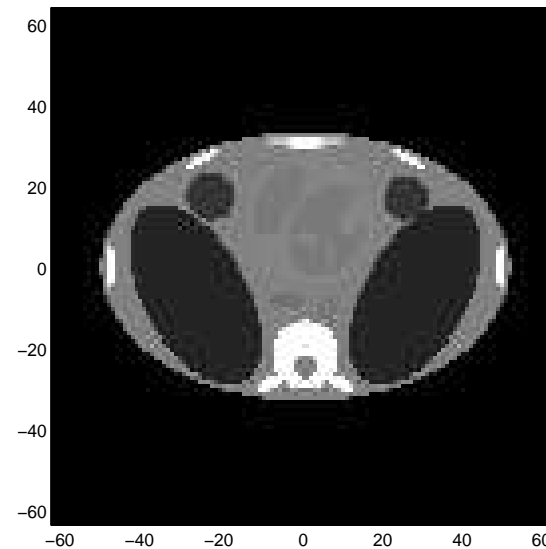
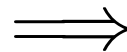
$$f(x, y) \rightarrow \boxed{\text{TR}} \rightarrow p(r, \phi)$$

$$p(r, \phi) = \int_{L_{r,\phi}} f(x, y) dl$$

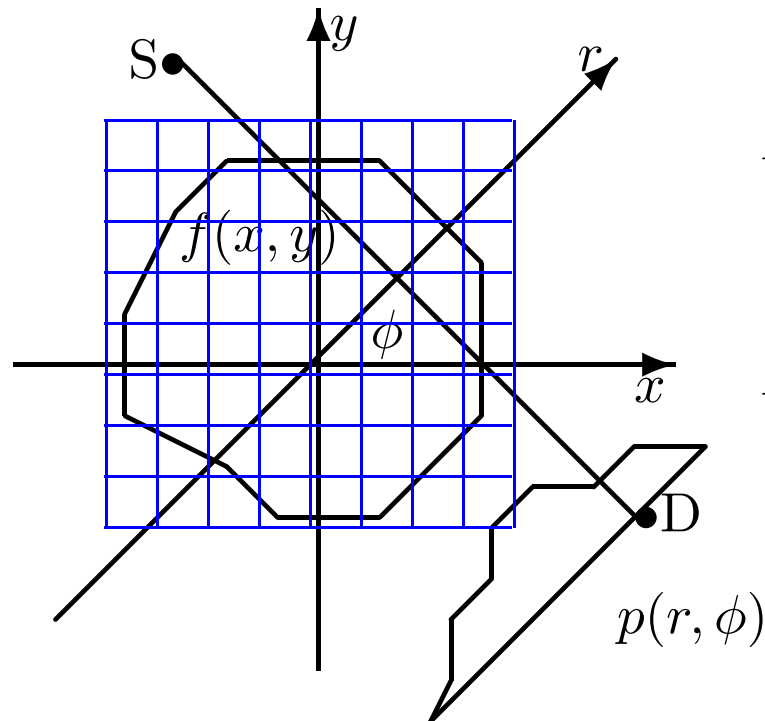
$$p(r, \phi) = \iint_D f(x, y) \delta(r - x \cos \phi - y \sin \phi) dx dy$$



?



## TOMOGRAPHIE X



$$p(r, \theta) = -\ln \left( \frac{I}{I_0} \right) = \int_L f(x, y) dl$$

$$p(r, \phi) = \iint_D f(x, y) \delta(r - x \cos \phi - y \sin \phi) dx dy$$

RADON:

$$f(x, y) = \left( -\frac{1}{2\pi^2} \right) \int_0^\pi \int_{-\infty}^{+\infty} \frac{\frac{\partial}{\partial r} p(r, \phi)}{(r - x \cos \phi - y \sin \phi)} dr d\phi$$

$$f(x, y) = \left( -\frac{1}{2\pi^2} \right) \int_0^\pi \int_{-\infty}^{+\infty} \frac{\frac{\partial}{\partial r} p(r, \phi)}{(r - x \cos \phi - y \sin \phi)} dr d\phi$$

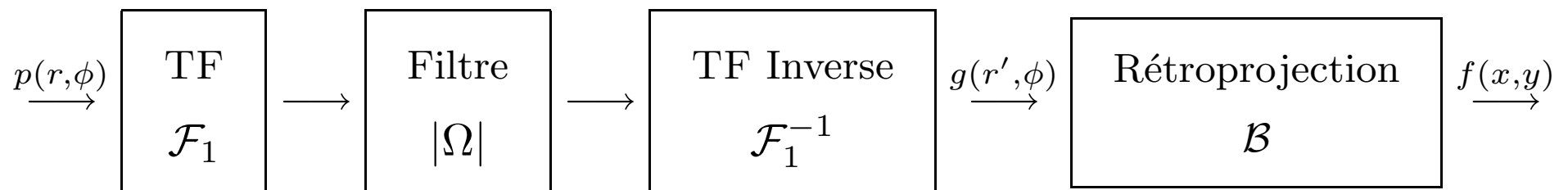
Dérivation  $\mathcal{D}$  :  $\bar{p}(r, \phi) = \frac{\partial p(r, \phi)}{\partial r}$

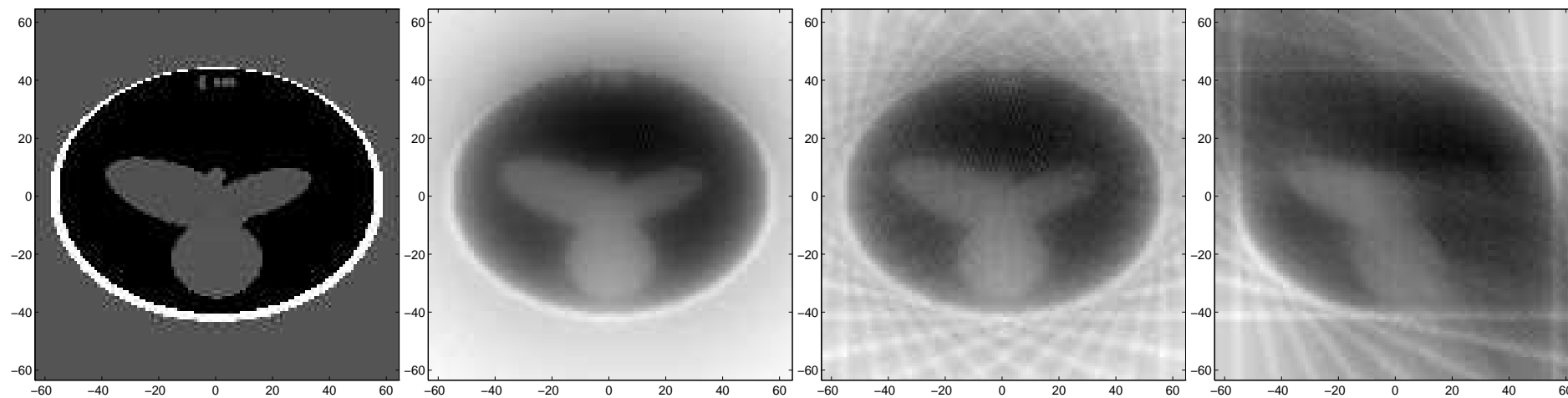
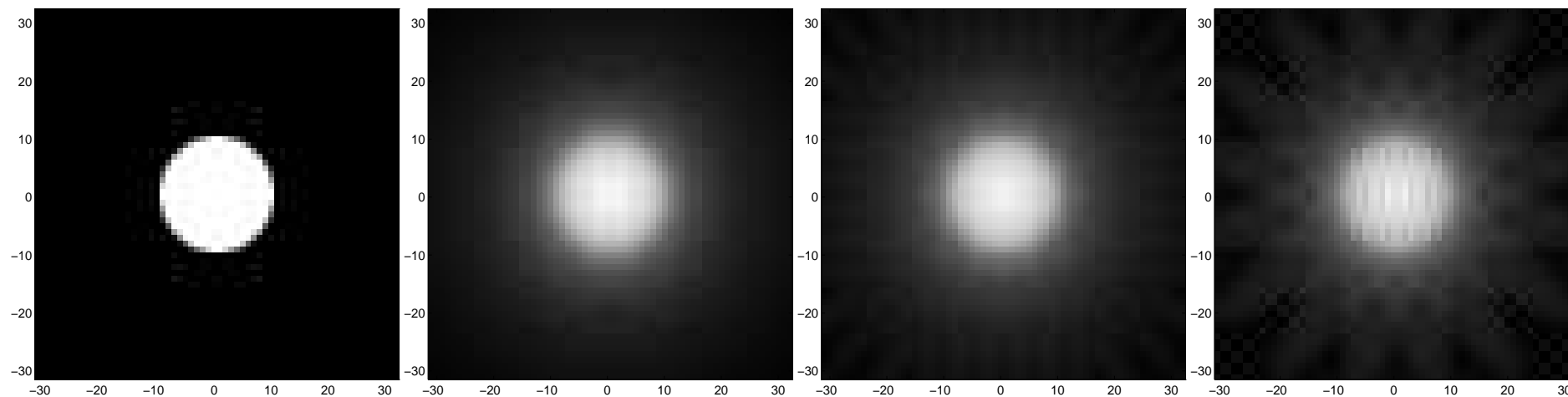
Transformée de Hilbert  $\mathcal{H}$  :  $g(r', \phi) = \frac{1}{\pi} \int_0^\infty \frac{\bar{p}(r, \phi)}{(r - r')} dr$

Rétroprojection  $\mathcal{B}$  :  $f(x, y) = \frac{1}{2\pi} \int_0^\pi g(r' = x \cos \phi + y \sin \phi, \phi) d\phi$

$$f = \mathcal{B} \mathcal{H} \mathcal{D} \mathcal{R} f$$

- Rétroprojection des projections filtrées :



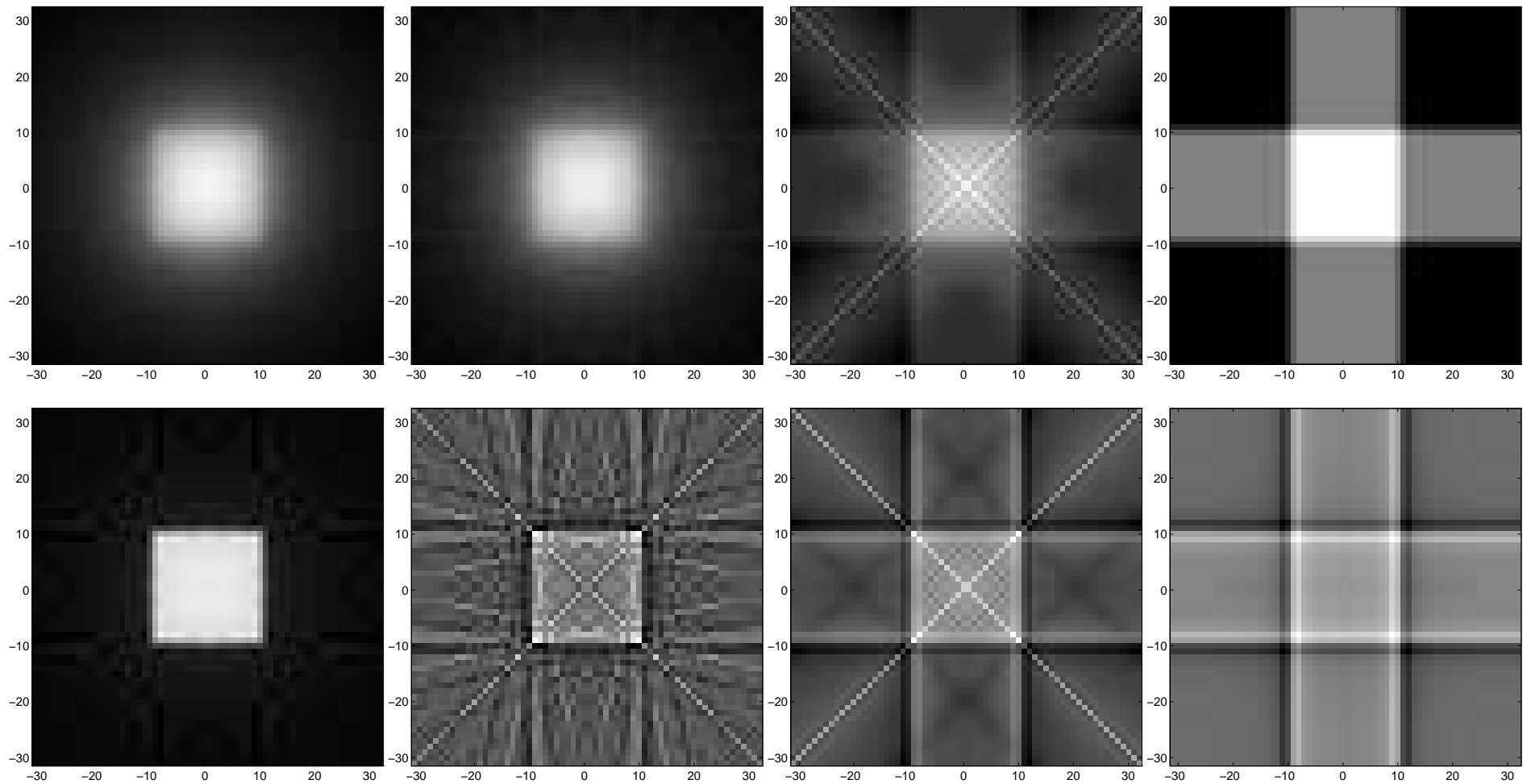


objets initiaux

64 projections

16 projections

8 projections  
entre 0 et  $\pi/2$ .



64 projections

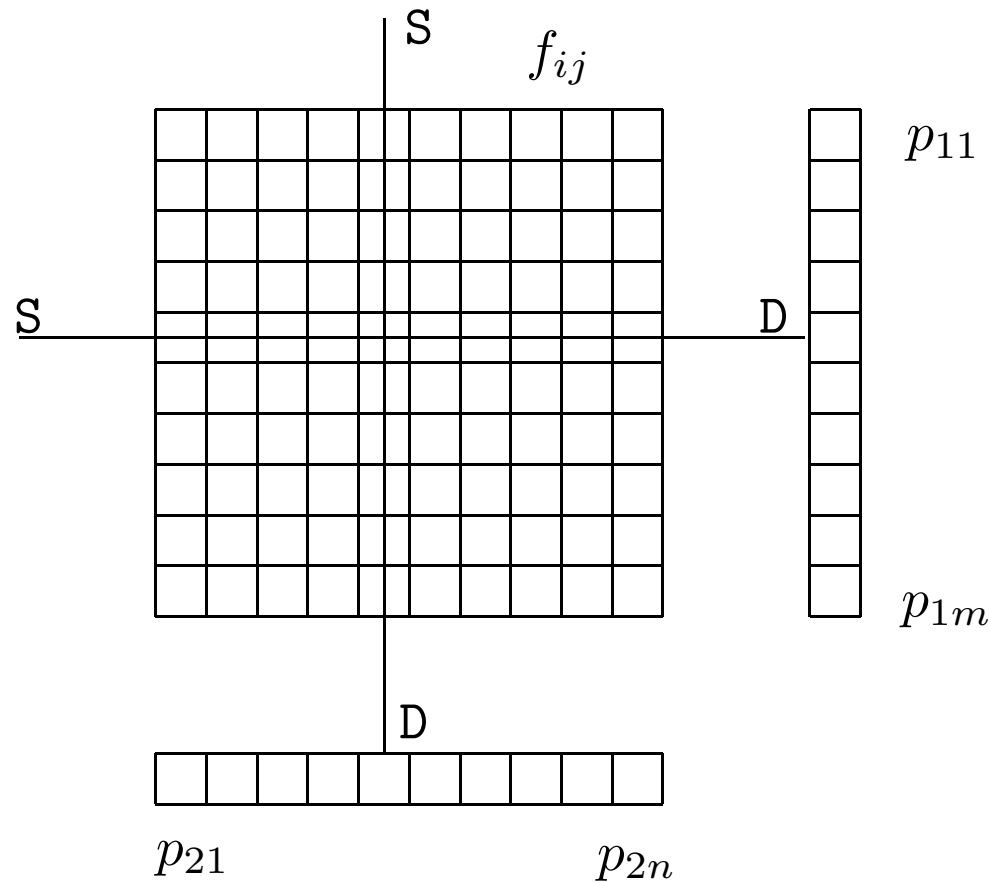
16 projections

4 projections

2 projections

## MÉTHODES D'INVERSION NAÏVES

### Reconstruction d'image en tomographie à partir de deux projections



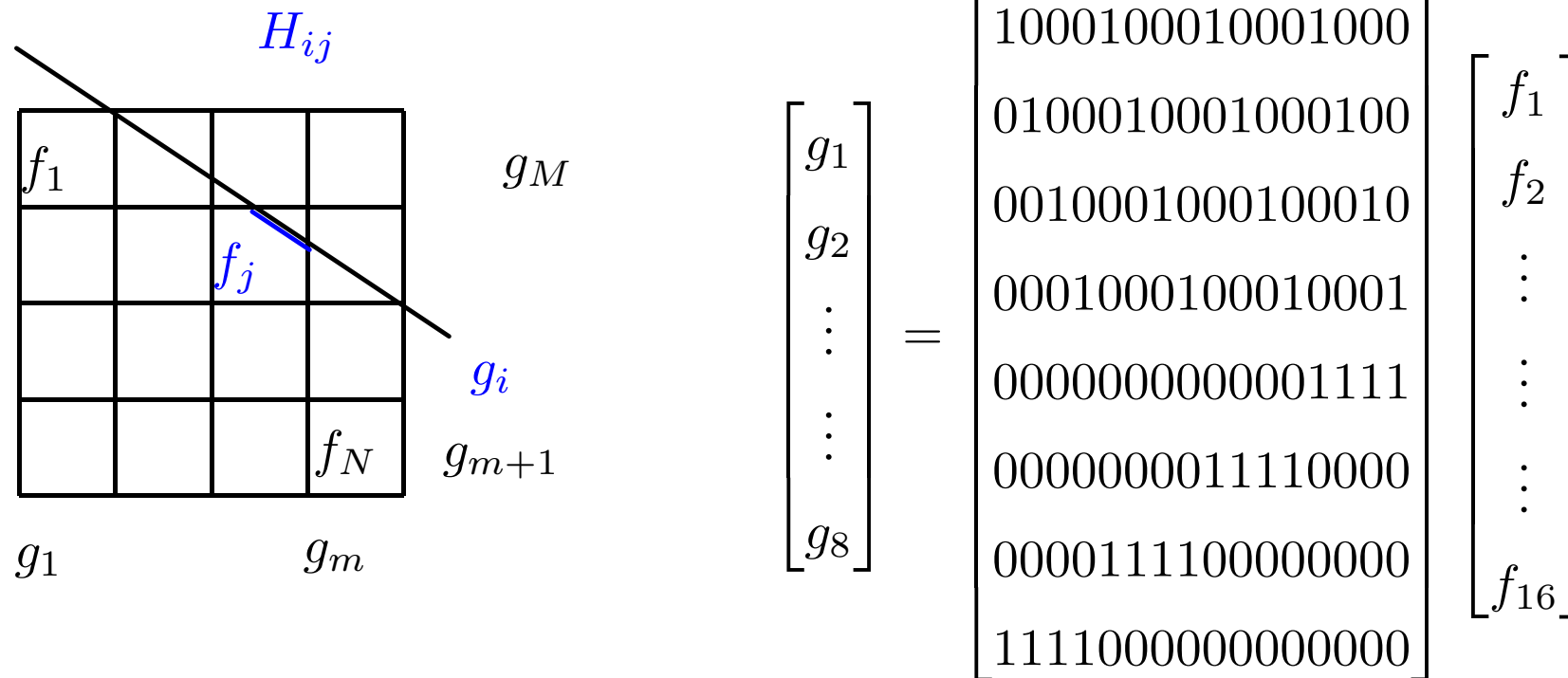
$p_{1i}$  projection verticale

$p_{2j}$  projection horizontale

$$\sum_{j=1}^n f_{ij} = p_{1i}, \quad i = 1, \dots, m$$

$$\sum_{i=1}^m f_{ij} = z_{2j}, \quad j = 1, \dots, n$$





$$g = Hf$$

- Problème direct: Étant donné  $f$  calculer  $g$
- Problème inverse: Étant donné  $g$  déterminer ou estimer  $f$
- Existence de la solution
- Unicité de la solution
- Stabilité de la solution (sensibilité vis-à-vis des perturbations)

## Existence et Unicité :

				40
		?		40
				40
				40
40	40	40	40	

10	10	10	10	40
10	10	10	10	40
10	10	10	10	40
10	10	10	10	40
40	40	40	40	

40	0	0	0	40
0	40	0	0	40
0	0	40	0	40
0	0	0	40	40
40	40	40	40	

15	5	5	15	40
5	15	15	5	40
5	15	15	5	40
15	5	5	15	40
40	40	40	40	

20	0	0	20	40
0	20	20	0	40
0	20	20	0	40
20	0	0	20	40
40	40	40	40	

20	10	5	5	40
10	20	5	5	40
5	5	20	10	40
5	5	10	20	40
40	40	40	40	

## Stabilité :

$$\mathbf{H} = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix} \quad \mathbf{H}^{-1} = \begin{bmatrix} 25 & -41 & 10 & -6 \\ -41 & 68 & -17 & 10 \\ 10 & -17 & 5 & -3 \\ -6 & 10 & -3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 23 \\ 33 \\ 31 \end{bmatrix} \longrightarrow \mathbf{f} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

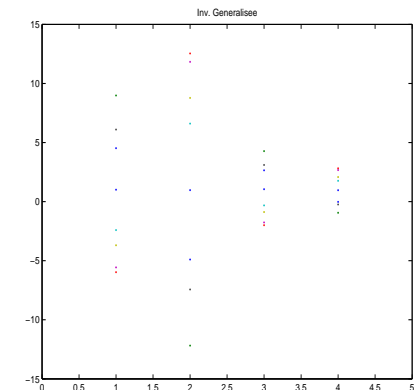
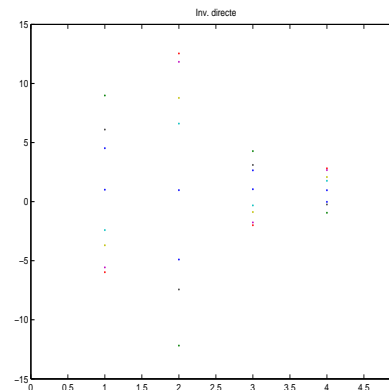
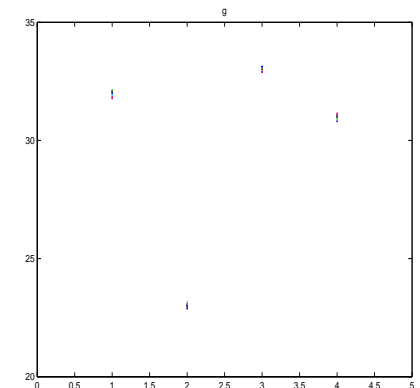
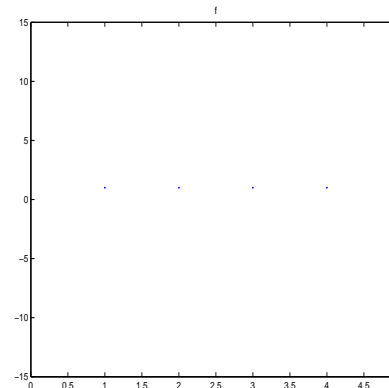
$$\begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix} \begin{bmatrix} f_1 + \delta f_1 \\ f_2 + \delta f_2 \\ f_3 + \delta f_3 \\ f_4 + \delta f_4 \end{bmatrix} = \begin{bmatrix} 32.1 \\ 22.9 \\ 33.1 \\ 30.9 \end{bmatrix} \longrightarrow \mathbf{f} + \delta \mathbf{f} = \begin{bmatrix} 9.2 \\ -12.6 \\ 4.5 \\ -1.1 \end{bmatrix}$$

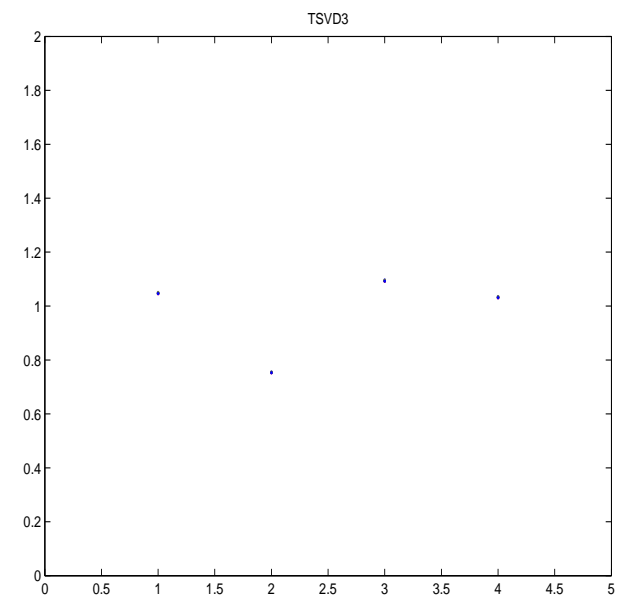
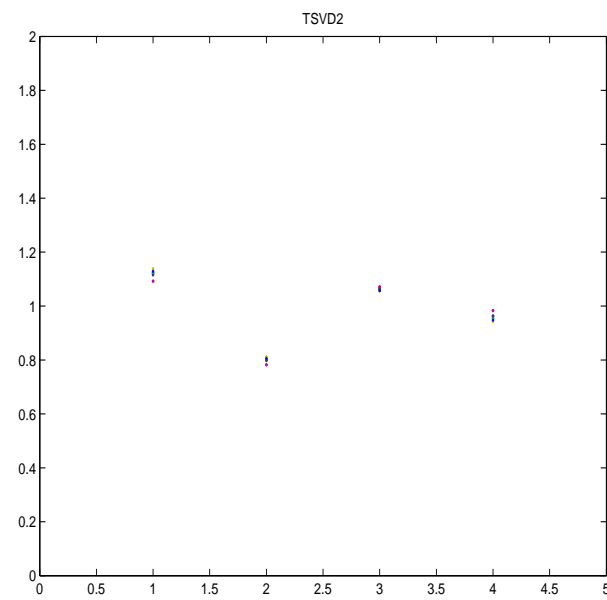
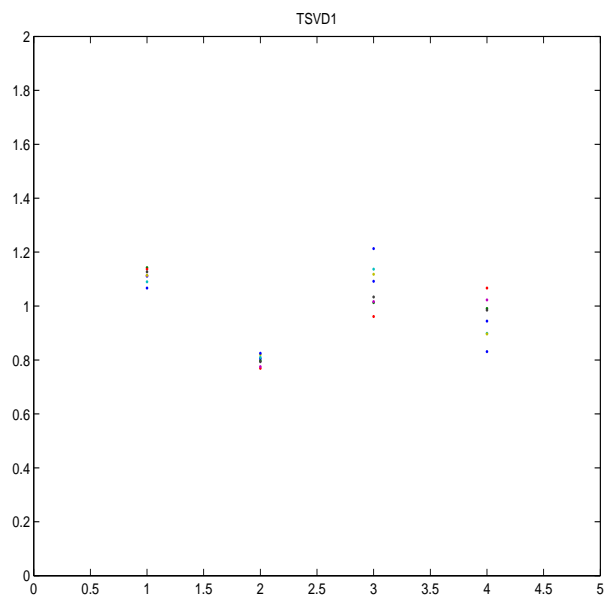
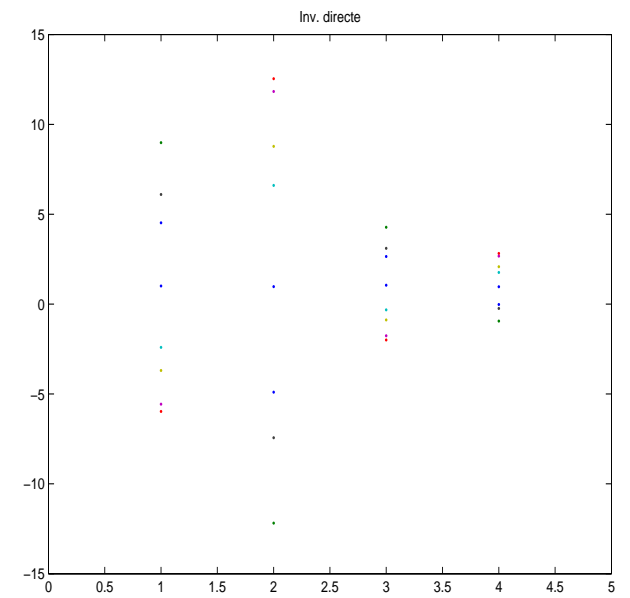
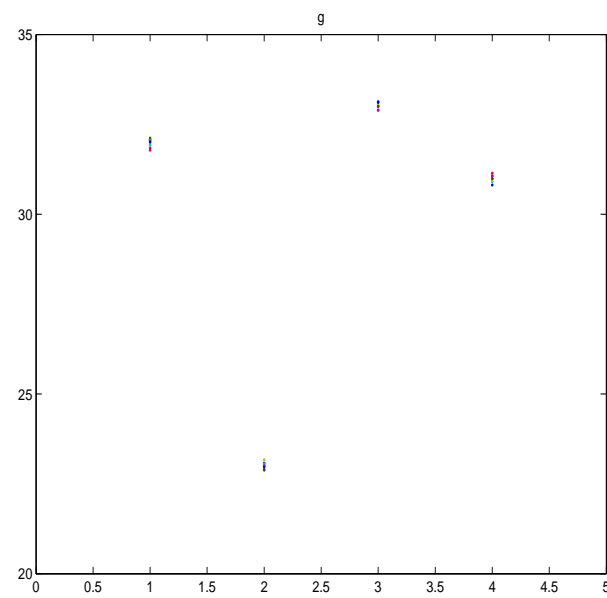
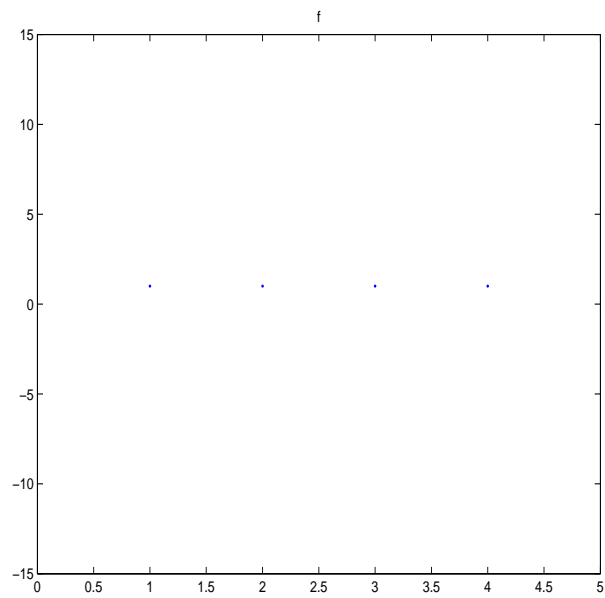
$$\frac{\|\delta \mathbf{g}\|}{\|\mathbf{g}\|} = \frac{1}{300} \longrightarrow \frac{\|\delta \mathbf{f}\|}{\|\mathbf{f}\|} = \frac{10}{1}$$

$$\frac{\|\delta \mathbf{f}\|}{\|\mathbf{f}\|} \leq \text{cond}(A) \frac{\|\delta \mathbf{g}\|}{\|\mathbf{g}\|}$$

$$\boldsymbol{\lambda} = \{30.3, 3.86, .84, 0.01\}$$

$$\longrightarrow \text{cond}(A) = \frac{\max |\lambda_i|}{\min |\lambda_i|} = \frac{30.3}{0.01} \sim 3000$$





Sensibilité aux erreurs sur les paramètres du modèle:

$$\begin{bmatrix} 10 & 7 & 8.1 & 7.2 \\ 7.08 & 5.04 & 6 & 5 \\ 8 & 5.98 & 9.89 & 9 \\ 6.99 & 4.99 & 99.98 & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 23 \\ 33 \\ 31 \end{bmatrix} \longrightarrow \mathbf{f} = \begin{bmatrix} -81 \\ 137 \\ -34 \\ 22 \end{bmatrix}$$

CONCLUSION:

Même si la solution exacte existe, il vaut mieux y renoncer et rechercher une solution approchée mais moins sensible aux erreurs

## MÉTHODES ALGÈBRIQUES DÉTERMINISTES

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon}$$

- Rétrojection ou filtre adapté :  $\hat{\mathbf{f}} = \mathbf{H}^t \mathbf{g}$ , mais  $\mathbf{H}^t \mathbf{H} \neq \mathbf{I}$
- Inversion directe lorsque possible:  $\hat{\mathbf{f}} = \mathbf{H}^{-1} \mathbf{g}$
- Inversion généralisée:
  - $M > N$  et  $\text{rang}\{A\} = N \longrightarrow \hat{\mathbf{f}} = (\mathbf{H}^t \mathbf{H})^{-1} \mathbf{H}^t \mathbf{g}$
  - $M < N$  et  $\text{rang}\{A\} = M \longrightarrow \hat{\mathbf{f}} = \mathbf{H}^t (\mathbf{H} \mathbf{H}^t)^{-1} \mathbf{g}$
  - Décomposition en valeurs singulières :  $\hat{\mathbf{f}} = \sum_{k=1}^r \frac{\langle \mathbf{u}_k, \mathbf{g} \rangle}{\lambda_k} \mathbf{v}_k$
- Restriction de l'espace des solutions :

min  $\Omega(\mathbf{f})$  s.c.  $\mathbf{H}\mathbf{f} = \mathbf{g}$ , avec, par exemple

$$\Omega(\mathbf{f}) = \sum_{j=1}^n f_j^2, \quad \Omega(\mathbf{f}) = - \sum_{j=1}^n f_j \log f_j, \quad \Omega(\mathbf{f}) = \sum_{j=1}^n \log f_j, \quad \dots$$

## RÉGULARISATION

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon}$$

- Moindre carrés :

$$\hat{\mathbf{f}} = \arg \min_{\mathbf{f}} \{Q(\mathbf{f}) = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2\} \longrightarrow \hat{\mathbf{f}} = (\mathbf{H}^t \mathbf{H})^{-1} \mathbf{H}^t \mathbf{g}$$

- Optimisation sous contraintes :

$$\hat{\mathbf{f}} = \arg \min_{\mathbf{f}} \{\Omega(\mathbf{f}) = \|\mathbf{f}\|^2\} \quad \text{s.c. } \mathbf{g} = \mathbf{H}\mathbf{f} \longrightarrow \hat{\mathbf{f}} = \mathbf{H}^t (\mathbf{H}\mathbf{H}^t)^{-1} \mathbf{g}$$

- Optimisation sous contraintes :

$$\min \Omega(\mathbf{f}) \quad \text{s.c. } \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 < \epsilon \quad \text{ou} \quad \min Q(\mathbf{f}) \quad \text{s.c. } \|\mathbf{f}\|^2 < \eta$$

- Critère régularisé :

$$\hat{\mathbf{f}} = \arg \min_{\mathbf{f}} \{\|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda \Omega(\mathbf{f})\}$$



## MÉTHODES PROBABILISTES

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon}$$

On se place dans le cadre de l'estimation :

- Maximum de vraisemblance (MV)

**Avantages :**

- Prise en compte explicite de la nature du bruit

$$p(\mathbf{g}|\mathbf{f}) = p_{\boldsymbol{\epsilon}}(\mathbf{g} - \mathbf{H}\mathbf{f}) \longrightarrow \hat{\mathbf{f}} = \arg \max_{\mathbf{f}} \{\ln p(\mathbf{g}|\mathbf{f})\}$$

Exemple: bruit gaussien  $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \sigma_{\boldsymbol{\epsilon}}^2 \mathbf{I}) \longrightarrow p(\mathbf{g}|\mathbf{f}) = \mathcal{N}(\mathbf{H}\mathbf{f}, \sigma_{\boldsymbol{\epsilon}}^2 \mathbf{I})$

$$p(\mathbf{g}|\mathbf{f}) \propto \exp \left[ -\frac{1}{\sigma_{\boldsymbol{\epsilon}}^2} \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 \right] \longrightarrow$$

$$\hat{\mathbf{f}} = \arg \max_{\mathbf{f}} \{p(\mathbf{g}|\mathbf{f})\} = \arg \min_{\mathbf{f}} \{\|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2\}$$

- Estimation bayésienne (EB)

**Avantages :**

- Prise en compte explicite de la nature du bruit
- Prise en compte d'information *a priori* sur la solution

## APPROCHE ESTIMATION BAYÉSIENNE

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon}$$

- Modèle d'observation + Hypothèse sur le bruit  $\longrightarrow p(\mathbf{g}|\mathbf{f}) = p_{\boldsymbol{\epsilon}}(\mathbf{g} - \mathbf{H}\mathbf{f})$
- Information *a priori*  $p(\mathbf{f})$
- Règle de Bayes : 
$$p(\mathbf{f}|\mathbf{g}) = \frac{p(\mathbf{g}|\mathbf{f}) p(\mathbf{f})}{p(\mathbf{g})}$$
- Choix d'un estimateur basé sur  $p(\mathbf{f}|\mathbf{g})$

### Lien avec régularisation :

- Estimation au sens du maximum *a posteriori* (MAP) :

$$\hat{\mathbf{f}} = \arg \max_{\mathbf{f}} \{p(\mathbf{f}|\mathbf{g})\} = \arg \max_{\mathbf{f}} \{p(\mathbf{g}|\mathbf{f}) p(\mathbf{f})\}$$

$$\hat{\mathbf{f}} = \arg \min_{\mathbf{f}} \{-\ln p(\mathbf{g}|\mathbf{f}) - \ln p(\mathbf{f})\}$$

avec  $Q(\mathbf{g}, \mathbf{H}\mathbf{f}) = -\ln p(\mathbf{g}|\mathbf{f})$  et  $\lambda\Omega(\mathbf{f}) = -\ln p(\mathbf{f})$

## CAS LINÉAIRE GAUSSIEN

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon}$$

- Hypothèse sur le bruit :

$$\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \sigma_{\epsilon}^2 \mathbf{I}) \longrightarrow \mathbf{g}|\mathbf{f} \sim \mathcal{N}(\mathbf{H}\mathbf{f}, \sigma_{\epsilon}^2 \mathbf{I}) \longrightarrow p(\mathbf{g}|\mathbf{f}) \propto \exp \left[ -\frac{1}{2\sigma_{\epsilon}^2} \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 \right]$$

- Hypothèse gaussienne sur  $\mathbf{f}$  :

$$\mathbf{f} \sim \mathcal{N}(\mathbf{f}_0, \sigma_f^2 (\mathbf{D}^t \mathbf{D})^{-1}) \longrightarrow p(\mathbf{f}) \propto \exp \left[ -\frac{1}{2\sigma_f^2} \|\mathbf{D}[\mathbf{f} - \mathbf{f}_0]\|^2 \right]$$

- loi a posteriori:

$$p(\mathbf{f}|\mathbf{g}) \propto \exp \left[ -\frac{1}{2\sigma_{\epsilon}^2} \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 - \frac{1}{2\sigma_f^2} \|\mathbf{D}[\mathbf{f} - \mathbf{f}_0]\|^2 \right]$$

- La solution au sens du MAP :  $\hat{\mathbf{f}} = \arg \max_{\mathbf{f}} \{p(\mathbf{f}|\mathbf{g})\} = \arg \min_{\mathbf{f}} \{J(\mathbf{f})\}$

$$\text{avec } J(\mathbf{f}) = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda \|\mathbf{D}(\mathbf{f} - \mathbf{f}_0)\|^2, \quad \lambda = \frac{\sigma_{\epsilon}^2}{\sigma_f^2}$$

- **Avantage** : caractéristique de la solution

$$\mathbf{f}|\mathbf{g} \sim \mathcal{N}(\hat{\mathbf{f}}, \hat{\mathbf{P}}) \text{ avec } \hat{\mathbf{f}} = \hat{\mathbf{P}}\mathbf{H}^t(\mathbf{g} - \mathbf{H}\mathbf{f}_0), \quad \hat{\mathbf{P}} = (\mathbf{H}^t\mathbf{H} + \lambda\mathbf{D}^t\mathbf{D})^{-1}$$

$$g_i = f + \epsilon_i, \quad i = 1, \dots, m \quad \longrightarrow \quad \mathbf{g} = \mathbf{1}f + \boldsymbol{\epsilon}$$

$$\boldsymbol{\epsilon} \sim \mathbf{N}(\mathbf{0}, \beta \mathbf{I}) \quad \longrightarrow \quad \mathbf{g}|f, \beta \sim \mathbf{N}(f\mathbf{1}, \beta \mathbf{I})$$

$$f|f_0, \theta \sim \mathbf{N}(f_0, \theta)$$

$$f|\mathbf{g}, \beta, f_0, \theta \sim \mathbf{N}(f_m, \theta_m),$$

$$\text{avec } \theta_m = \left(\frac{1}{\theta} + \frac{m}{\beta}\right)^{-1} = \beta \left(\frac{\beta}{\theta} + m\right)^{-1}$$

$$\text{et } f_m = \theta_m \left(\frac{f_0}{\theta} + m \frac{\bar{y}}{\beta}\right) = \frac{\theta_m}{\beta} \left(\frac{\beta}{\theta} f_0 + m \bar{g}\right)$$

$$\bar{g} = \frac{1}{m} \sum_i g_i$$

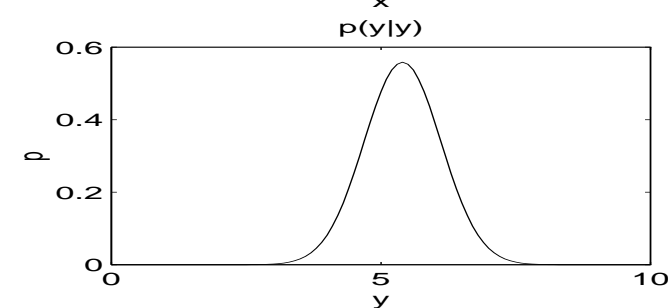
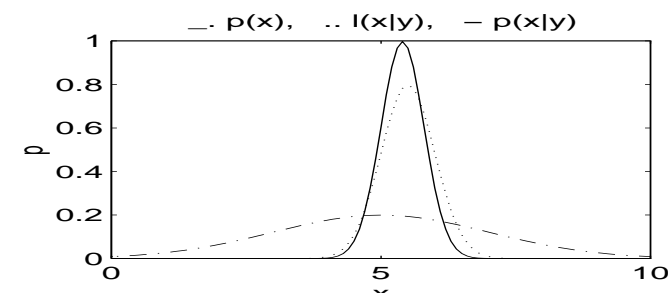
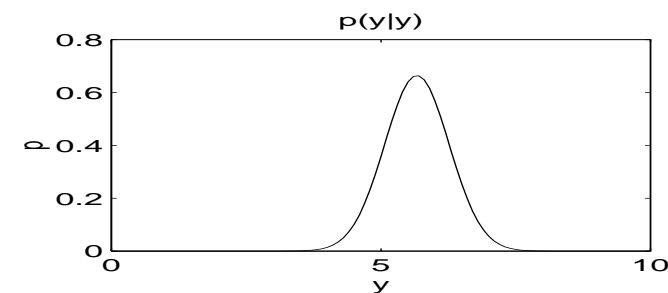
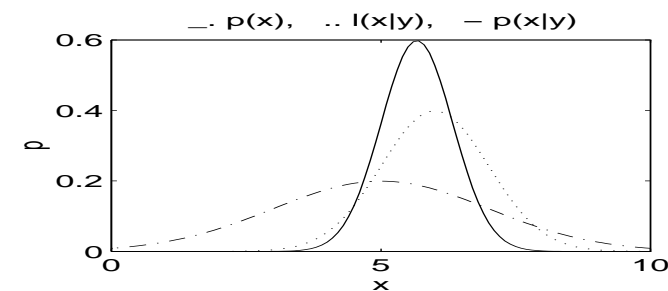
$$\beta = 0 \quad \text{ou} \quad \theta \longrightarrow \infty \quad \text{ou} \quad m \longrightarrow \infty \quad \longrightarrow \quad f_m = \bar{g}$$

$$\beta \longrightarrow \infty \quad \text{ou} \quad \theta = 0 \quad \text{ou} \quad m = 0 \quad \longrightarrow \quad f_m = f_0$$

*Exemples :*  $\beta = 1, f_0 = 5, \theta = 2$

*Ex1 :*  $m = 1, g = 6, \longrightarrow f_m = 5.6, \theta_m = 3$

*Ex2 :*  $m = 2, \mathbf{g} = [6, 5], \bar{g} = 5.5 \longrightarrow f_m = 5.4, \theta_m = .4$



## SIGNAUX CONTINUS / IMAGES CONTINUES

$$\mathbf{g} = \mathbf{H} \mathbf{f} + \boldsymbol{\epsilon}$$

$\mathbf{f}$  Image continue : **Modèle de Gauss-Markov**

$$\mathbf{f} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_f)$$

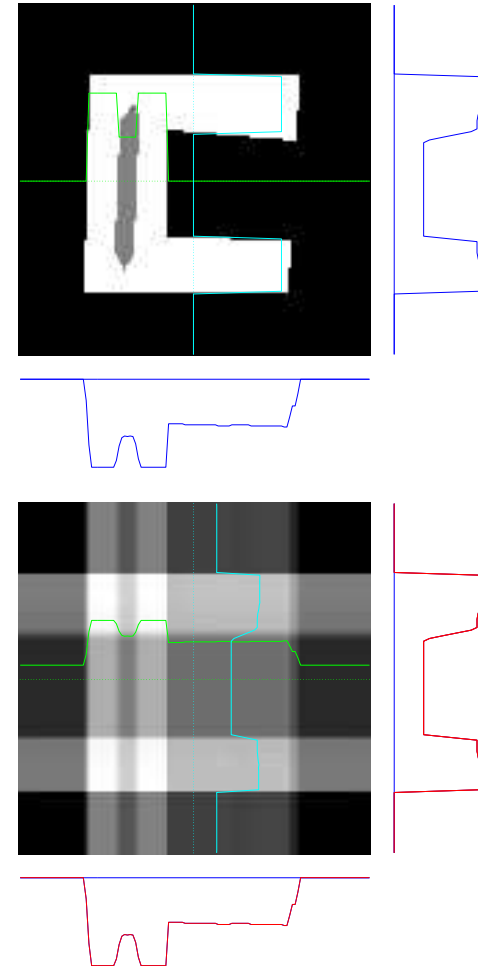
$$p(f_j | f_i, i \neq j) = \mathcal{N}(\beta f_{j-1}, \sigma_f^2)$$

$$p(f(\mathbf{r}) | f(\mathbf{s})) = \mathcal{N}\left(\beta \sum_{\mathbf{s} \in \mathcal{V}(\mathbf{r})} f(\mathbf{s}), \sigma_f^2\right)$$

MAP :

$$J(\mathbf{f}) = \|\mathbf{g} - \mathbf{H} \mathbf{f}\|^2 + \sum_j (f_j - \beta f_{j-1})^2$$

$$J(\mathbf{f}) = \|\mathbf{g} - \mathbf{H} \mathbf{f}\|^2 + \sum_{\mathbf{r}} \left(f(\mathbf{r}) - \beta \sum_{\mathbf{s} \in \mathcal{V}(\mathbf{r})} f(\mathbf{s})\right)^2$$



# SIGNAUX CONTINUS PAR MORCEAUX / IMAGES CONTINUES PAR RÉGIONS

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon},$$

$\mathbf{f}$  Image continue par région :

Champs de Markov intensité-contours

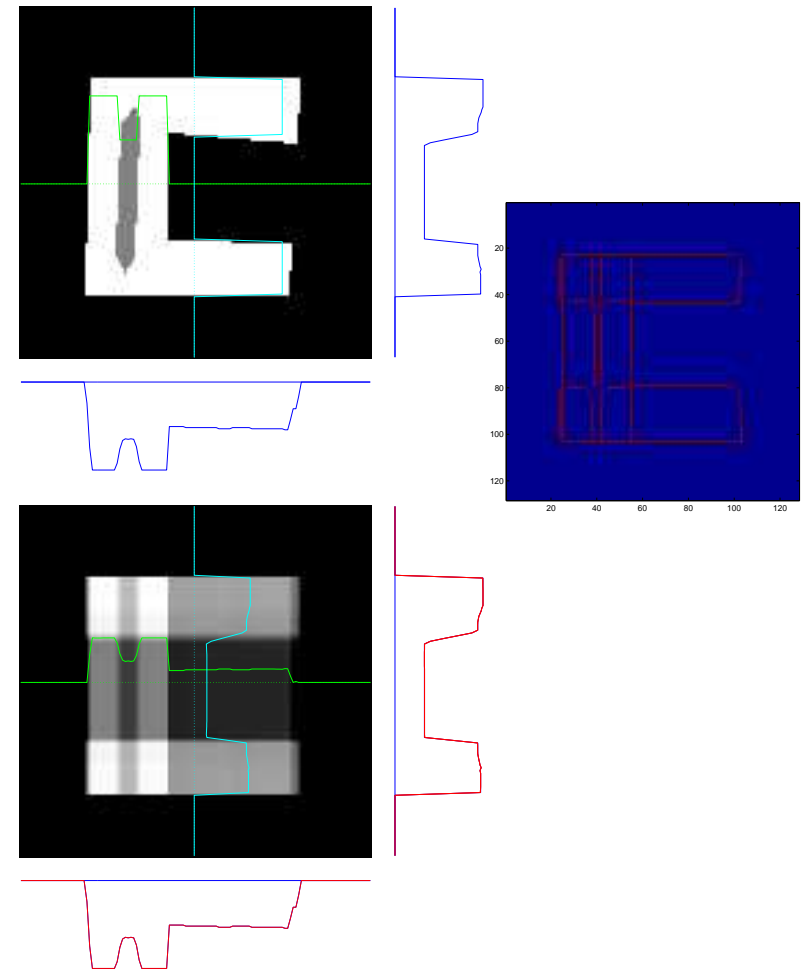
Variable cachée contour:  $q(\mathbf{r})$

$$p(f_j|q_j, f_i, i \neq j) = \mathcal{N}(\beta(1 - q_j)f_{j-1}, \sigma_f^2)$$

$$p(f(\mathbf{r})|q(\mathbf{r}), f(\mathbf{s})) \\ = \mathcal{N}\left(\beta(1 - q(\mathbf{r})) \sum_{\mathbf{s} \in \mathcal{V}(\mathbf{r})} f(\mathbf{s}), \sigma_f^2\right)$$

MAP :

$$J(\mathbf{f}) = \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 \\ + \sum_{\mathbf{r}} (1 - q(\mathbf{r})) \left( f(\mathbf{r}) - \beta \sum_{\mathbf{s} \in \mathcal{V}(\mathbf{r})} f(\mathbf{s}) \right)^2$$



## IMAGERIE D'OBJETS COMPOSÉS D'UN NOMBRE FINI DE MATÉRIAUX

$$g = Hf + \epsilon$$

$f$  représente une image d'un objet  $f(\mathbf{r})$   
composé de plusieurs matières homogènes

Modèle: Champs de Markov Intensité-régions

Introduction d'une variable label  $z(\mathbf{r})$

$$z(\mathbf{r}) = k, \quad k = 1, \dots, K$$

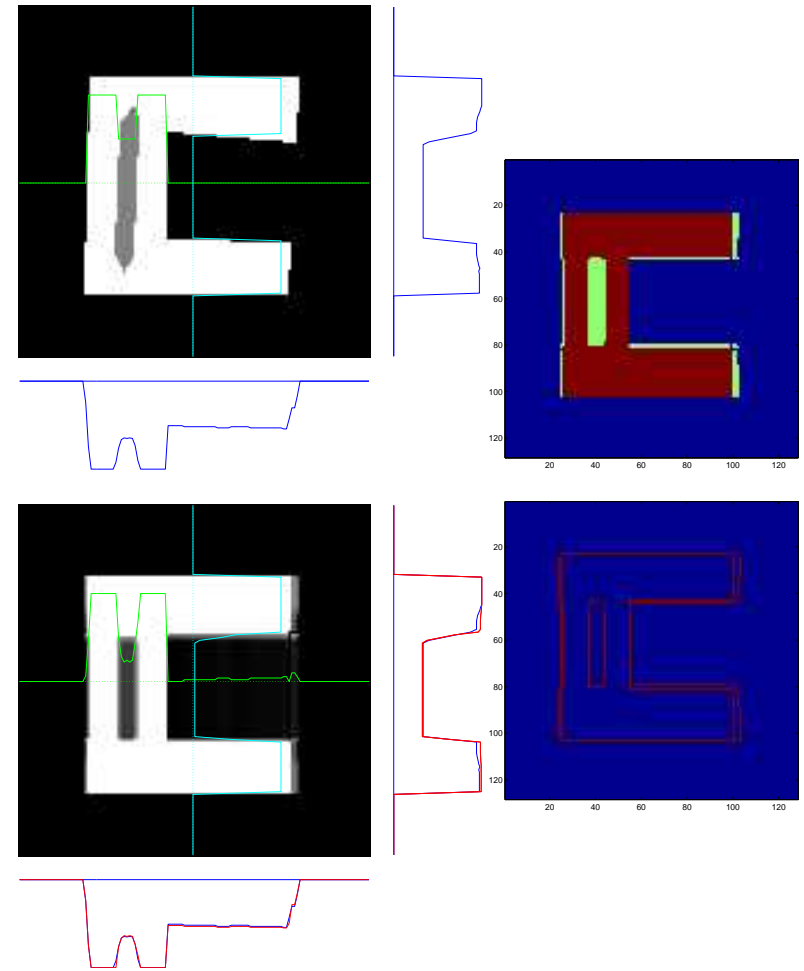
$$\mathcal{R}_k = \{\mathbf{r} : z(\mathbf{r}) = k\}, \quad \mathcal{R} = \cup_k \mathcal{R}_k$$

$$p(f(\mathbf{r})|z(\mathbf{r}) = k) = \mathcal{N}(f(\mathbf{r})|m_k, \sigma_k^2)$$

$\mathbf{z} = \{z(\mathbf{r}), \mathbf{r} \in \mathcal{R}\}$  une image segmentée

Champs de Potts:

$$p(\mathbf{z}) \propto \exp \left[ \alpha \sum_{\mathbf{r} \in \mathcal{R}} \sum_{\mathbf{s} \in \mathcal{V}(\mathbf{r})} \delta(z(\mathbf{r}) - z(\mathbf{s})) \right]$$



- Modèle de mélange (IGM) avec  $\{z(\mathbf{r}), \mathbf{r} \in \mathcal{R}\}$  i.i.d.

$$p(\mathbf{z}) = \prod_{k=1}^K p_k \quad \text{avec} \quad P(z(\mathbf{r}) = k) = p_k \quad \text{et} \quad \sum_{k=1}^K p_k = 1$$

- Modèle de mélange avec un champs de Markov (Potts) pour  $\mathbf{z}$

$$p(\mathbf{z}) \propto \exp \left[ \alpha \sum_{\mathbf{r} \in \mathcal{R}} \sum_{\mathbf{s} \in \mathcal{V}(\mathbf{r})} \delta(z(\mathbf{r}) - z(\mathbf{s})) \right]$$

- Lois *a priori* pour les hyperparamètres  $\boldsymbol{\theta} = \{\sigma_\epsilon^2, (m_k, \sigma_k^2), k = 1, \dots, K\}$  :

$$\begin{aligned} p(m_k) &= \mathcal{N}(m_k | m_{k0}, \sigma_{k0}^2), & p(\sigma_k^2) &= \mathcal{IG}(\sigma_k^2 | \alpha_{k0}, \beta_{k0}), \\ p(\boldsymbol{\Sigma}_k) &= \mathcal{IW}(\boldsymbol{\Sigma}_k | \alpha_{k0}, \boldsymbol{\Lambda}_{k0}), & p(\sigma_{\epsilon_i}^2) &= \mathcal{IG}(\sigma_{\epsilon_i}^2 | \alpha_0^{\epsilon_i}, \beta_0^{\epsilon_i}). \end{aligned}$$

- Loi *a posteriori* jointe de  $\mathbf{f}$ ,  $\mathbf{z}$  and  $\boldsymbol{\theta}$

$$p(\mathbf{f}, \mathbf{z}, \boldsymbol{\theta} | \mathbf{g}) \propto p(\mathbf{g} | \mathbf{f}, \boldsymbol{\theta}_1) p(\mathbf{f} | \mathbf{z}, \boldsymbol{\theta}_2) p(\mathbf{z} | \boldsymbol{\theta}_2) p(\boldsymbol{\theta})$$



Supposons d'abord  $\theta$  et  $z$  connus.

Alors l'estimée (MAP) devient:

$$\hat{\mathbf{f}} = \arg \max_{\mathbf{f}} \{p(\mathbf{f}|\mathbf{g}, z, \theta)\} = \arg \min_{\mathbf{f}} \{J(\mathbf{f}|\mathbf{g}, z, \theta)\}.$$

- Si les pixels dans une région supposés i.i.d. :

$$\begin{aligned} J(\mathbf{f}|\mathbf{g}, z, \theta) &= \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda \sum_{k=1}^K \frac{\|\mathbf{f}_k - m_k \mathbf{1}\|^2}{\sigma_k^2} \\ &= \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda \sum_{k=1}^K \sum_{\mathbf{r} \in \mathcal{R}_k} \frac{\|f(\mathbf{r}) - m_k\|^2}{\sigma_k^2} \end{aligned}$$

- Si un modèle markovien pour les pixels (dépendance locale) :

$$\begin{aligned} J(\mathbf{f}|\mathbf{g}, z, \theta) &= \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda \sum_{k=1}^K \frac{\|\tilde{\mathbf{f}}_k - m_k \mathbf{1}\|^2}{\sigma_k^2} \\ &= \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 + \lambda \sum_{k=1}^K \frac{1}{\sigma_k^2} \left( \tilde{f}(\mathbf{r}) - \beta_{\mathbf{r}} \sum_{\mathbf{s} \in (\mathcal{V}(\mathbf{r}) \cap \mathcal{R}_k)} \tilde{f}(\mathbf{s}) \right)^2 \end{aligned}$$

où  $\tilde{f}(\mathbf{r}) = f(\mathbf{r}) - m(\mathbf{r})$ ,  $\beta_{\mathbf{r}} = \frac{1}{n_{\mathbf{r}}}$ ,  $n_{\mathbf{r}} = \text{Card}(\mathcal{V}(\mathbf{r}) \cap \mathcal{R}_k)$ .

Mais, on peut estimer  $(\mathbf{f}, \mathbf{z}, \boldsymbol{\theta})$  en utilisant  $p(\mathbf{f}, \mathbf{z}, \boldsymbol{\theta}|\mathbf{g})$

- MAP (Algorithme 1):

$$\left\{ \begin{array}{l} \hat{\mathbf{f}} = \arg \max_{\mathbf{f}} \{p(\mathbf{f}|\mathbf{z}, \boldsymbol{\theta}, \mathbf{g})\} \\ \hat{\boldsymbol{\theta}} = \arg \max_{\boldsymbol{\theta}} \{p(\boldsymbol{\theta}|\mathbf{f}, \mathbf{z}, \mathbf{g})\} \quad \text{ou} = \arg \max_{\boldsymbol{\theta}} \{p(\boldsymbol{\theta}|\mathbf{z}, \mathbf{g})\} \\ \hat{\mathbf{z}} = \arg \max_{\mathbf{z}} \{p(\mathbf{z}|\mathbf{f}, \boldsymbol{\theta}, \mathbf{g})\} \quad \text{ou} = \arg \max_{\mathbf{z}} \{p(\mathbf{z}|\boldsymbol{\theta}, \mathbf{g})\} \end{array} \right.$$

- MAP-Gibbs (Algorithme 2):

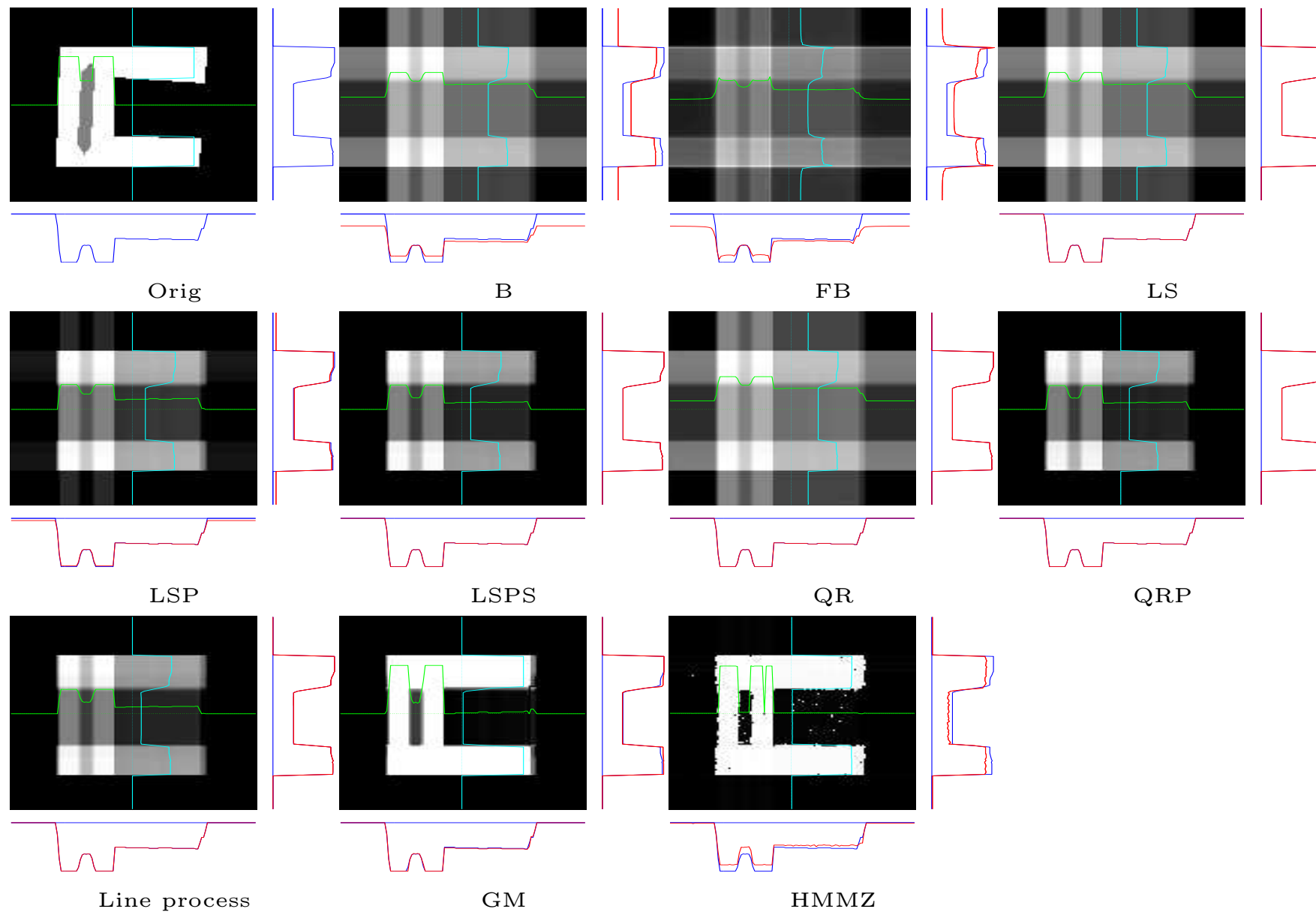
$$\left\{ \begin{array}{l} \hat{\mathbf{f}} = \arg \max_{\mathbf{f}} \{p(\mathbf{f}|\mathbf{z}, \boldsymbol{\theta}, \mathbf{g})\} \\ \text{échantillonner } \hat{\boldsymbol{\theta}} \text{ suivant } p(\boldsymbol{\theta}|\mathbf{f}, \mathbf{z}, \mathbf{g}) \quad \text{ou suivant } p(\boldsymbol{\theta}|\mathbf{z}, \mathbf{g}) \\ \text{échantillonner } \hat{\mathbf{z}} \text{ suivant } p(\mathbf{z}|\mathbf{f}, \boldsymbol{\theta}, \mathbf{g}) \quad \text{ou suivant } p(\mathbf{z}|\boldsymbol{\theta}, \mathbf{g}) \end{array} \right.$$

- MAP-Gibbs (Algorithme 3):

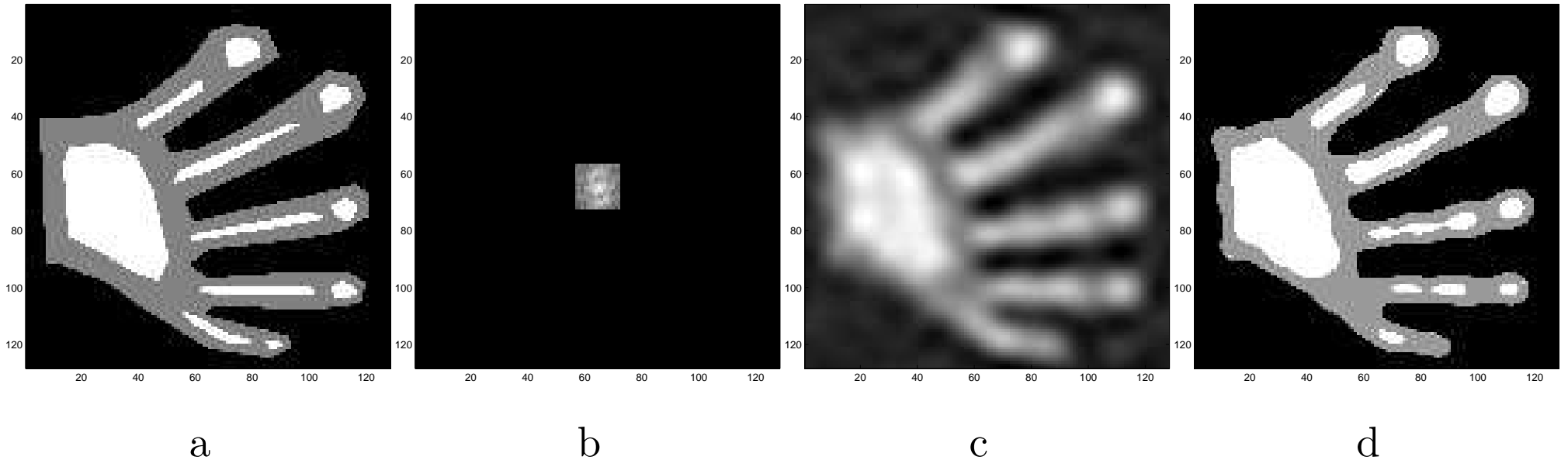
$$\left\{ \begin{array}{l} \hat{\mathbf{f}} = \arg \max_{\mathbf{f}} \{p(\mathbf{f}|\mathbf{z}, \boldsymbol{\theta}, \mathbf{g})\} \\ \hat{\boldsymbol{\theta}} = \arg \max_{\boldsymbol{\theta}} \{p(\boldsymbol{\theta}|\mathbf{f}, \mathbf{z}, \mathbf{g})\} \quad \text{or} = \arg \max_{\boldsymbol{\theta}} \{p(\boldsymbol{\theta}|\mathbf{z}, \mathbf{g})\} \\ \text{échantillonner } \hat{\mathbf{z}} \text{ suivant } p(\mathbf{z}|\mathbf{f}, \boldsymbol{\theta}, \mathbf{g}) \quad \text{ou suivant } p(\mathbf{z}|\boldsymbol{\theta}, \mathbf{g}) \end{array} \right.$$

## EXPRESSION DES LOIS *a posteriori*

$$\left\{ \begin{array}{l}
 p(\mathbf{f}|\mathbf{z}, \boldsymbol{\theta}, \mathbf{g}) = \mathcal{N}(\mathbf{f}|\hat{\mathbf{f}}, \hat{\boldsymbol{\Sigma}}) \quad \text{avec} \\
 \hat{\boldsymbol{\Sigma}} = (\mathbf{H}^t \boldsymbol{\Sigma}_\epsilon^{-1} \mathbf{H} + \boldsymbol{\Sigma}_z^{-1})^{-1} \quad \text{et} \quad \hat{\mathbf{f}} = \hat{\boldsymbol{\Sigma}} (\mathbf{H}^t \boldsymbol{\Sigma}_\epsilon^{-1} \mathbf{g} + \boldsymbol{\Sigma}_z^{-1} \mathbf{m}_z) \\
 \\
 p(\mathbf{z}|\mathbf{g}, \boldsymbol{\theta}) \propto p(\mathbf{g}|\mathbf{z}, \boldsymbol{\theta}) p(\mathbf{z}) \quad \text{avec} \\
 p(\mathbf{g}|\mathbf{z}, \boldsymbol{\theta}) = \mathcal{N}(\mathbf{g}|\mathbf{H} \mathbf{m}_z, \boldsymbol{\Sigma}_g) \quad \text{avec} \quad \boldsymbol{\Sigma}_g = \mathbf{H} \boldsymbol{\Sigma}_z \mathbf{H}^t + \boldsymbol{\Sigma}_\epsilon \\
 \\
 p(m_k|\mathbf{z}, \mathbf{f}) = \mathcal{N}(m_k|\mu_k, v_k) \quad \text{avec} \quad v_k = \left( \frac{n_k}{\sigma_k^2} + \frac{1}{\sigma_{k0}^2} \right)^{-1} \quad \text{et} \quad \mu_k = v_k \left( \frac{n_k \bar{f}_k}{\sigma_k^2} + \frac{m_{k0}}{\sigma_{k0}^2} \right) \\
 \\
 p(\sigma_k^2|\mathbf{f}, \mathbf{z}) = \mathcal{IG}(\sigma_k^2|\alpha_k, \beta_k) \quad \text{avec} \quad \alpha_k = \alpha_{k0} + \frac{n_k}{2} \quad \text{et} \quad \beta_k = \beta_{k0} + \frac{n_k \bar{s}_k}{2} \\
 \text{où} \quad \bar{f}_k = \frac{1}{n_k} \sum_{\mathbf{r} \in \mathcal{R}_k} f_i(\mathbf{r}) \quad \text{et} \quad \bar{s}_k = \sum_{\mathbf{r} \in \mathcal{R}_k} (f(\mathbf{r}) - m_k)^2 \\
 \\
 p(\sigma_\epsilon^2|\mathbf{f}, \mathbf{g}) = \mathcal{IG}(\sigma_\epsilon^2|\alpha^\epsilon, \beta^\epsilon) \quad \text{avec} \quad \alpha^\epsilon = \frac{n}{2} + \alpha_0^\epsilon \quad \text{et} \quad \beta^\epsilon = \frac{1}{2} \|\mathbf{g} - \mathbf{H} \mathbf{f}\|^2 + \beta_0^\epsilon \\
 \\
 n_k = \text{nombre de pixels dans } \mathcal{R}_k \quad \text{et} \quad n = \text{nombre total de pixels.}
 \end{array} \right.$$

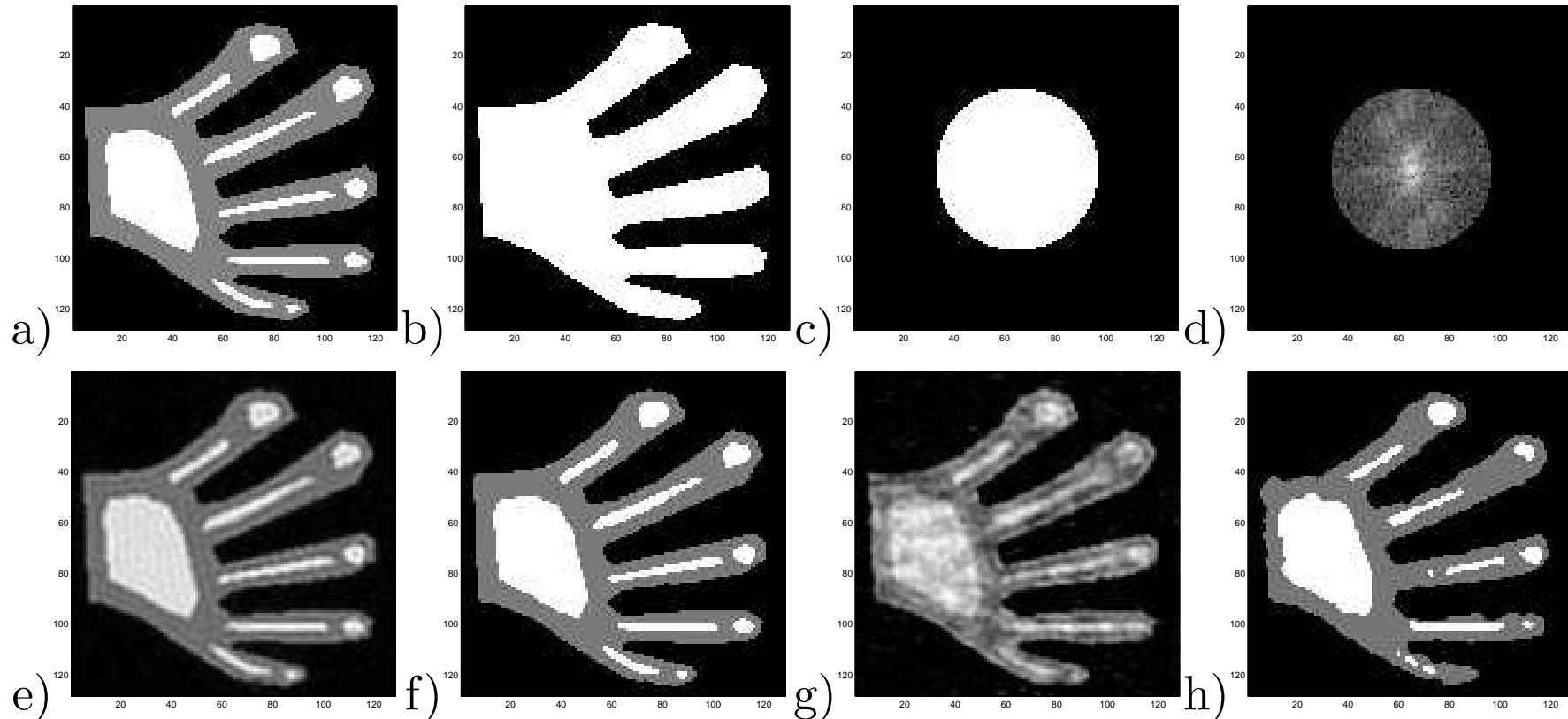


## SIMULATIONS



- a) Objet (une image de  $128 \times 128$  pixels),
- b) Spectre observé (seul les valeurs de la TF sur une grille de  $16 \times 16$  sont observées),
- c) Reconstruction par TF inverse,
- d) Reconstruction par la méthode proposée

## SIMULATIONS (2)



a) objet, b) support connu, c) support des données, d) données mesurées,

Résultats avec phases connues : e) TF inverse et f) méthode proposée,

Résultats avec phase inconnues : g) Gerchberg-Saxton et h) méthode proposée.