Sources separation methods: An overview

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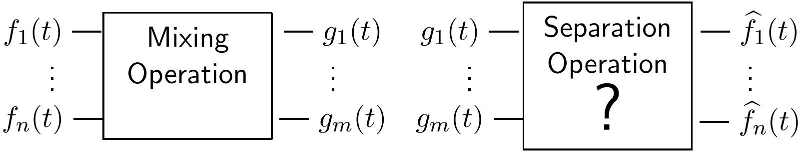
Introduction

Mixing models and separation:



- General Linear Mixing Model :

$$\boldsymbol{g}(t) = \int \boldsymbol{A}(t, t')\boldsymbol{f}(t') \, \mathrm{d}t'$$

- Convolutional Mixing Model:

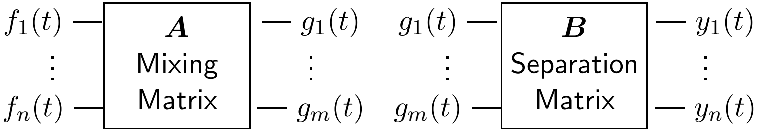
$$\boldsymbol{g}(t) = \int \boldsymbol{A}(t - t')\boldsymbol{f}(t') \, \mathrm{d}t'$$

- Instantaneous Mixing Model:

$$\boldsymbol{g}(t) = \boldsymbol{A}\boldsymbol{f}(t)$$

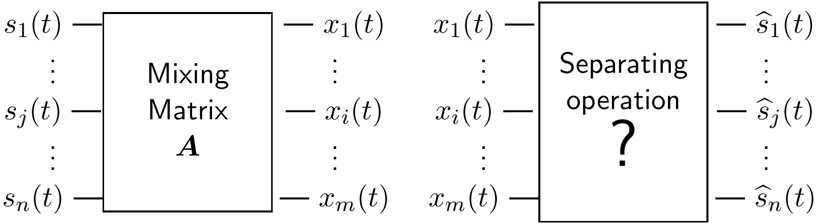
Introduction

Convolutional Mixing $\longrightarrow$ **Multi Chanel Deconvolution**
Instantaneous Mixing $\longrightarrow$ **Source Separation**



- Undeterminations: $\longrightarrow \boldsymbol{B} = \boldsymbol{P}\,\boldsymbol{\Lambda}\,\boldsymbol{A}^{-1}$
  where $\boldsymbol{P}$ is a Permutation matrix and $\boldsymbol{\Lambda}$ a scale (diagonal) matrix.
- Main Hypothesis:: $f_1(t), \ldots, f_n(t)$ are:
  non correlated (PCA) or
  independents (ICA).
- Classical methods : Infomax, Contrast function based, Higher Ordre Sup., Maximum Likelihood, Bayesian Approach

Introduction



Signals:

$$x_i(t) = \sum_{j=1}^{n} A_{i,j} s_j(t) + \epsilon_i(t), \quad t \in \mathcal{T}, \qquad i = 1, \cdots, m$$

Images:

$$x_i(\boldsymbol{r}) = \sum_{j=1}^{n} A_{i,j} s_j(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}), \ \boldsymbol{r} \in \mathcal{R}, \qquad i = 1, \cdots, m$$

General source separation problem

$$x_i(t) = \sum_{j=1}^{N} A_{ij} s_j(t) + \epsilon_i(t), \quad i = 1, \cdots, M$$

$$\boldsymbol{x}(t) = \boldsymbol{A}\,\boldsymbol{s}(t) + \boldsymbol{\epsilon}(t), \quad t = t_1, \cdots, t_T$$

$$\boldsymbol{X} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_M(t) \end{bmatrix} \quad \boldsymbol{E} = \begin{bmatrix} \epsilon_1(t) \\ \epsilon_2(t) \\ \vdots \\ \epsilon_M(t) \end{bmatrix} \quad \boldsymbol{S} = \begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_N(t) \end{bmatrix}$$

$$\boldsymbol{X} = \boldsymbol{A}\,\boldsymbol{S} + \boldsymbol{E}$$

Extension for images

$$x_i(\boldsymbol{r}) = \sum_{j=1}^{N} A_{ij} s_j(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}), \quad \boldsymbol{r} = (x, y) \in R^2$$

- $\boldsymbol{A}$: Mixing matrix, Loading matrix
- $\boldsymbol{s}(t)$: sources, factors (principales, independent), codebook, ...
- $\boldsymbol{x}(t)$: observations, mixtures, data

General source separation problem

$$g_i(t) = \sum_{j=1}^{N} A_{ij} f_j(t) + \epsilon_i(t), \quad i = 1, \cdots, M$$

$$\boldsymbol{g}(t) = \boldsymbol{A}\, \boldsymbol{f}(t) + \boldsymbol{\epsilon}(t), \quad t = t_1, \cdots, t_T$$

$$\boldsymbol{G} = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_M(t) \end{bmatrix} \quad \boldsymbol{E} = \begin{bmatrix} \epsilon_1(t) \\ \epsilon_2(t) \\ \vdots \\ \epsilon_M(t) \end{bmatrix} \quad \boldsymbol{F} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_N(t) \end{bmatrix}$$

$$\boldsymbol{G} = \boldsymbol{A}\, \boldsymbol{F} + \boldsymbol{E}$$

Extension for images

$$g_i(\boldsymbol{r}) = \sum_{j=1}^{N} A_{ij} f_j(\boldsymbol{r}) + \epsilon_i(\boldsymbol{r}), \quad \boldsymbol{r} = (x, y) \in R^2$$

- $\boldsymbol{A}$: Mixing matrix, Loading matrix
- $\boldsymbol{f}(t)$: sources, factors (principales, independent), codebook, ...
- $\boldsymbol{g}(t)$: observations, mixtures, data

General source separation problem

$$\boldsymbol{g}(t) = \boldsymbol{A}\boldsymbol{f}(t) + \boldsymbol{\epsilon}(t), \quad t \in [1, \cdots, T]$$

$$\boldsymbol{g}(\boldsymbol{r}) = \boldsymbol{A}\boldsymbol{f}(\boldsymbol{r}) + \boldsymbol{\epsilon}(\boldsymbol{r}), \quad \boldsymbol{r} = (x, y) \in R^2$$

- $\boldsymbol{f}$ unknown sources
- $\boldsymbol{A}$ mixing matrix, $\boldsymbol{a}_{*j}$ steering vectors
- $\boldsymbol{g}$ observed signals
- $\boldsymbol{\epsilon}$ represents the errors of modeling and measurement

$$\boldsymbol{g} = \boldsymbol{A}\boldsymbol{f} \longrightarrow g_i = \sum_j a_{ij} f_j \longrightarrow \boldsymbol{g} = \sum_j \boldsymbol{a}_{*j} f_j$$

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_1 & 0 & f_2 & 0 \\ 0 & f_1 & 0 & f_2 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix}$$

$$\boldsymbol{g} = \boldsymbol{A}\boldsymbol{f} = \boldsymbol{F}\boldsymbol{a} \quad \text{with} \quad \boldsymbol{F} = \boldsymbol{f} \odot \boldsymbol{I}, \quad \boldsymbol{a} = \text{vec}(\boldsymbol{A})$$

- $\boldsymbol{A}$ known, estimation of $\boldsymbol{f}$: $\quad \boldsymbol{g} = \boldsymbol{A}\boldsymbol{f} + \boldsymbol{\epsilon}$
- $\boldsymbol{f}$ known, estimation of $\boldsymbol{A}$: $\quad \boldsymbol{g} = \boldsymbol{F}\boldsymbol{a} + \boldsymbol{\epsilon}$
- Joint estimation of $\boldsymbol{f}$ and $\boldsymbol{A}$: $\quad \boldsymbol{g} = \boldsymbol{A}\boldsymbol{f} + \boldsymbol{\epsilon} = \boldsymbol{F}\boldsymbol{a} + \boldsymbol{\epsilon}$

Deterministic methods

$$\boldsymbol{g} = \boldsymbol{A}\boldsymbol{f} \quad \text{or} \quad \boldsymbol{G} = \boldsymbol{A}\boldsymbol{F}$$

Matrix factorization, Sources separation, Compressed sensing

- $\boldsymbol{A}$ known, find $\boldsymbol{f}$

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \left\{ \|\boldsymbol{g} - \boldsymbol{A}\boldsymbol{f}\|^2 + \lambda \|\boldsymbol{f}\|^2 \right\} = (\boldsymbol{A}'\boldsymbol{A} + \lambda \boldsymbol{I})^{-1} \boldsymbol{A}'\boldsymbol{g}$$

- $\boldsymbol{f}$ known, find $\boldsymbol{A}$

$$\widehat{\boldsymbol{A}} = \arg\min_{\boldsymbol{A}} \left\{ \|\boldsymbol{g} - \boldsymbol{A}\boldsymbol{f}\|^2 + \lambda \|\boldsymbol{A}\|^2 \right\} = \boldsymbol{g}\boldsymbol{f}'(\boldsymbol{f}\boldsymbol{f}' + \lambda \boldsymbol{I})^{-1}$$

- Both $\boldsymbol{A}$ and $\boldsymbol{f}$ are unknown

$$(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{A}}) = \arg\min_{(\boldsymbol{f}, \boldsymbol{A})} \left\{ \|\boldsymbol{g} - \boldsymbol{A}\boldsymbol{f}\|^2 + \lambda_1 \|\boldsymbol{f}\|^2 + \lambda_2 \|\boldsymbol{A}\|^2 \right\}$$

Alternate optimisation

Deterministic methods

- Both $\boldsymbol{A}$ and $\boldsymbol{f}$ are unknown:

$$(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{A}}) = \arg \min_{(\boldsymbol{f},\boldsymbol{A})} \left\{ \|\boldsymbol{g} - \boldsymbol{A}\boldsymbol{f}\|^2 + \lambda_1 \|\boldsymbol{f}\|^2 + \lambda_2 \|\boldsymbol{A}\|^2 \right\}$$

- Undeterminations:
  - Permutation: $\boldsymbol{A}\boldsymbol{P}, \boldsymbol{P}'\boldsymbol{f}$
  - Scale: $k\boldsymbol{A}, \frac{1}{k}\boldsymbol{f}$
- Alternate optimisation

$$\begin{cases} \widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \left\{ \|\boldsymbol{g} - \boldsymbol{A}\boldsymbol{f}\|^2 + \lambda_1 \|\boldsymbol{f}\|^2 \right\} = (\boldsymbol{A}'\boldsymbol{A} + \lambda_1 \boldsymbol{I})^{-1} \boldsymbol{A}'\boldsymbol{g} \\ \widehat{\boldsymbol{A}} = \arg\min_{\boldsymbol{A}} \left\{ \|\boldsymbol{g} - \boldsymbol{A}\boldsymbol{f}\|^2 + \lambda_2 \|\boldsymbol{A}\|^2 \right\} = \boldsymbol{g}\boldsymbol{f}'(\boldsymbol{f}\boldsymbol{f}' + \lambda_2 \boldsymbol{I}) \end{cases}$$

- Importance of initialization and other constraintes such as positivity
  - Non-negative Matrix decomposition

Source separation: PCA or ICA approach

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{s} \longrightarrow \widehat{\boldsymbol{s}} = \boldsymbol{B}\boldsymbol{x}$$

- $\boldsymbol{A}$ mixing matrix
- $\boldsymbol{B}$ separation (demixing) matrix: $\boldsymbol{B} = \boldsymbol{A}^{-1} \longrightarrow \widehat{\boldsymbol{s}} = \boldsymbol{s}$
- Find $\boldsymbol{B}$ such that the components (sources) $\widehat{\boldsymbol{s}}$ be
  - Uncorrelated: Principal Components Analysis (PCA)
  - Independent: Independent Components Analysis (ICA)

- PCA: $\boldsymbol{x} = \boldsymbol{A}\boldsymbol{s} \longrightarrow \text{cov}[\boldsymbol{x}] = \boldsymbol{A}\text{cov}[\boldsymbol{s}]\boldsymbol{A}'$
  - Estimate $\text{cov}[\boldsymbol{x}] = \frac{1}{T}\sum_t(\boldsymbol{x}(t) - \bar{\boldsymbol{x}})(\boldsymbol{x}'(t) - \bar{\boldsymbol{x}})$
  - SVD decomposition: $\text{cov}[\boldsymbol{x}] = \boldsymbol{U}\Lambda\boldsymbol{U}'$
  - Identify: $\widehat{\boldsymbol{A}} = \boldsymbol{U}$, $\text{cov}[\boldsymbol{s}] = \boldsymbol{\Lambda} \longrightarrow \widehat{\boldsymbol{s}} = \Lambda^{1/2}\widehat{\boldsymbol{A}}'\boldsymbol{x}$
  - Uniqueness ?
    $\widehat{\boldsymbol{A}} = \boldsymbol{R}\boldsymbol{B}$ is also a solution for any rotational matrix $\boldsymbol{R}$.

Source separation: ICA approach

- ICA: Sources are supposed to be independent
- $\boldsymbol{x} = \boldsymbol{A}\boldsymbol{s} \longrightarrow p_x(\boldsymbol{x}) = |\boldsymbol{A}|-1\, p_s(\boldsymbol{A}\boldsymbol{s})$
- $\boldsymbol{B} = \boldsymbol{A}^{-1} \longrightarrow \widehat{\boldsymbol{s}} = \boldsymbol{B}\boldsymbol{x} = \boldsymbol{s} \longrightarrow p_s(\boldsymbol{B}\boldsymbol{x}) = |\boldsymbol{B}|^{-1}\, p_s(\boldsymbol{x})$

- Independence criteria:
  - Entropy: maximize the entropy $H = -\int p(\boldsymbol{s}) \ln p(\boldsymbol{s})\, \mathrm{d}\boldsymbol{s}$
  - Infomax: $KL\left(\prod_j p_j(\hat{s}_j) : p(\widehat{\boldsymbol{s}})\right)$
- $KL\left(\prod_j p_j([\boldsymbol{B}\boldsymbol{x}]_j) : p(\boldsymbol{B}\boldsymbol{x})\right)$ is a function of $\boldsymbol{B}$
- Minimization with respect to $\boldsymbol{B}$ gives ICA algorithms.

General Bayesian source separation problem

$$p(\boldsymbol{f}, \boldsymbol{A} | \boldsymbol{g}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) = \frac{p(\boldsymbol{g} | \boldsymbol{f}, \boldsymbol{A}, \boldsymbol{\theta}_1) \, p(\boldsymbol{f} | \boldsymbol{\theta}_2) \, p(\boldsymbol{A} | \boldsymbol{\theta}_3)}{p(\boldsymbol{g} | \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)}$$

- $p(\boldsymbol{g} | \boldsymbol{f}, \boldsymbol{A}, \boldsymbol{\theta}_1)$ likelihood
- $p(\boldsymbol{f} | \boldsymbol{\theta}_2)$ and $p(\boldsymbol{A} | \boldsymbol{\theta}_3)$ priors
- $p(\boldsymbol{f}, \boldsymbol{A} | \boldsymbol{g}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)$ joint posterior
- $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)$ hyper-parameters

Two approaches:

- Estimate first $\boldsymbol{A}$ and then use it for estimating $\boldsymbol{f}$
- Joint estimation

In real application, we also have to estimate $\boldsymbol{\theta}$:

$$p(\boldsymbol{f}, \boldsymbol{A}, \boldsymbol{\theta} | \boldsymbol{g}) = \frac{p(\boldsymbol{g} | \boldsymbol{f}, \boldsymbol{A}, \boldsymbol{\theta}_1) \, p(\boldsymbol{f} | \boldsymbol{\theta}_2) \, p(\boldsymbol{A} | \boldsymbol{\theta}_3) \, p(\boldsymbol{\theta})}{p(\boldsymbol{g})}$$

Bayesian inference for sources $\boldsymbol{f}$ when $\boldsymbol{A}$ is known

$$\boldsymbol{g} = \boldsymbol{A}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Prior knowledge on $\boldsymbol{\epsilon}$:

$$\boldsymbol{\epsilon} \sim \mathcal{N}(\boldsymbol{\epsilon}|0, v_\epsilon \boldsymbol{I}) \longrightarrow p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{A}) = \mathcal{N}(\boldsymbol{g}|\boldsymbol{A}\boldsymbol{f}, v_\epsilon \boldsymbol{I}) \propto \exp\left\{\frac{1}{2v_\epsilon}\|\boldsymbol{g} - \boldsymbol{A}\boldsymbol{f}\|^2\right\}$$

- Simple prior models for $\boldsymbol{f}$: $\quad p(\boldsymbol{f}|\alpha) \propto \exp\left\{-\alpha\|\boldsymbol{f}\|^2\right\}$
- Expression of the posterior law:

$$p(\boldsymbol{f}|\boldsymbol{g}, \boldsymbol{A}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{A})\, p(\boldsymbol{f}) \propto \exp\left\{-\frac{1}{2v_\epsilon} J(\boldsymbol{f})\right\}$$

with $\qquad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{A}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{f}\|^2, \quad \lambda = v_\epsilon \alpha$

- Link between MAP estimation and regularization

$$\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g}, \boldsymbol{A})\} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\}$$

- Solution: $\widehat{\boldsymbol{f}} = (\boldsymbol{A}'\boldsymbol{A} + \lambda \boldsymbol{I})^{-1}\boldsymbol{A}'\boldsymbol{g}$

Bayesian inference for sources $\boldsymbol{f}$ when $\boldsymbol{A}$ is known

- More general prior model $p(\boldsymbol{f}) \propto \exp\{-\alpha\Omega(\boldsymbol{f})\}$
- MAP:

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{A}\boldsymbol{f}\|^2 + \lambda\Omega(\boldsymbol{f}), \quad \lambda = v_\epsilon \alpha$$

$$\boxed{p(\boldsymbol{f}|\boldsymbol{\theta}, \boldsymbol{g})} \longrightarrow \boxed{\begin{array}{c}\text{Optimization of} \\ J(\boldsymbol{f}) = \frac{1}{2v_\epsilon}\|\boldsymbol{g} - \boldsymbol{A}\boldsymbol{f}\|^2 + \alpha\Omega(\boldsymbol{f})\end{array}} \longrightarrow \widehat{\boldsymbol{f}}$$

- Different priors=Different expressions for $\Omega(\boldsymbol{f})$
- Solution can be obtained using appropriate optimisation algorithm.

MAP estimation with sparsity enforcing priors

- Gaussian: $\Omega(\boldsymbol{f}) = \|\boldsymbol{f}\|^2 = \sum_j |f_j|^2$

$$J(\boldsymbol{f}) = \frac{1}{2v_\epsilon}\|\boldsymbol{g} - \boldsymbol{A}\boldsymbol{f}\|^2 + \alpha\|\boldsymbol{f}\|^2 \longrightarrow \widehat{\boldsymbol{f}} = [\boldsymbol{A}'\boldsymbol{A} + \lambda\boldsymbol{I}]^{-1}\boldsymbol{A}'\boldsymbol{g}$$

- Generalized Gaussian:

$$\Omega(\boldsymbol{f}) = \gamma \sum_j |f_j|^\beta)$$

- Student-t model:

$$\Omega(\boldsymbol{f}) = \frac{\nu+1}{2} \sum_j \log\left(1 + f_j^2/\nu\right)$$

- Elastic Net model:

$$\Omega(\boldsymbol{f}) = \sum_j \left[\gamma_1 |f_j| + \gamma_2 f_j^2\right]$$

For an extended list of such sparsity enforcing priors see:
A. Mohammad-Djafari, "Bayesian approach with prior models which enforce sparsity in signal and image processing," EURASIP Journal on Advances in Signal Processing, vol. Special issue on Sparse Signal Processing, 2012.

Estimation of $\boldsymbol{A}$ when the sources $\boldsymbol{f}$ are known

Source separation is a bilinear model:

$$\boldsymbol{g} = \boldsymbol{A}\boldsymbol{f} = \boldsymbol{F}\boldsymbol{a} = \boldsymbol{A}\boldsymbol{f}$$

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_1 & 0 & f_2 & 0 \\ 0 & f_1 & 0 & f_2 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix}$$

$$\boldsymbol{F} = \boldsymbol{f} \odot \boldsymbol{I}, \quad \boldsymbol{a} = \text{vec}(\boldsymbol{A})$$

- Problem is more ill-posed (underdetermined).
- We need absolutely to impose constraintes on elements or the structure of $\boldsymbol{A}$, for example:
  - Positivity of the elements
  - Toeplitz or TBT structure
  - Symmetry
  - Sparsity
- The same Bayesian approach then can be applied.

Estimation of $\boldsymbol{A}$ when the sources $\boldsymbol{f}$ are known

$$\boldsymbol{g} = \boldsymbol{A}\boldsymbol{f} + \boldsymbol{\epsilon} = \boldsymbol{F}\boldsymbol{a} + \boldsymbol{\epsilon}$$

- Prior on noise:

$$p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{A}) = \mathcal{N}(\boldsymbol{g}|\boldsymbol{A}\boldsymbol{f}, v_\epsilon \boldsymbol{I}) \quad \begin{aligned} &\propto \exp\left\{\tfrac{1}{2v_\epsilon}\|\boldsymbol{g}-\boldsymbol{A}\boldsymbol{f}\|^2\right\} \\ &\propto \exp\left\{\tfrac{1}{2v_\epsilon}\|\boldsymbol{g}-\boldsymbol{F}\boldsymbol{a}\|^2\right\}\end{aligned}$$

- Simple prior models for $\boldsymbol{a}$:

$$p(\boldsymbol{A}|\alpha) \propto \exp\left\{-\alpha\|\boldsymbol{a}\|^2\right\} \propto \exp\left\{-\alpha\|\boldsymbol{A}\|^2\right\}$$

- Expression of the posterior law:

$$p(\boldsymbol{A}|\boldsymbol{g},\boldsymbol{f}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{A})\, p(\boldsymbol{A}) \propto \exp\left\{-J(\boldsymbol{A})\right\}$$

$$\text{with} \qquad J(\boldsymbol{A}) = \frac{1}{2v_\epsilon}\|\boldsymbol{g}-\boldsymbol{A}\boldsymbol{f}\|^2 + \alpha\|\boldsymbol{A}\|^2$$

- MAP estimation:

$$\widehat{\boldsymbol{a}} = (\boldsymbol{F}'\boldsymbol{F} + \lambda\boldsymbol{I})^{-1}\boldsymbol{F}'\boldsymbol{g} \leftrightarrow \widehat{\boldsymbol{A}} = \boldsymbol{g}\boldsymbol{f}'(\boldsymbol{f}\boldsymbol{f}' + \lambda\boldsymbol{I})^{-1}$$

Bayesian source separation: both $\boldsymbol{A}$ and $\boldsymbol{f}$ unknown

$$p(\boldsymbol{f},\boldsymbol{A}|\boldsymbol{g},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)=\frac{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{A},\boldsymbol{\theta}_1)\,p(\boldsymbol{f}|\boldsymbol{\theta}_2)\,p(\boldsymbol{A}|\boldsymbol{\theta}_3)}{p(\boldsymbol{g}|\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)}$$

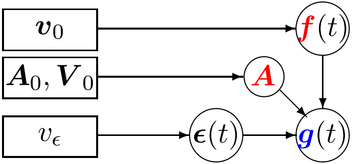
Two approaches:

- Joint estimation
- Estimate first $\boldsymbol{A}$ and then use it for estimating $\boldsymbol{f}$
- Joint estimation (JMAP):
  $(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{A}})=\arg\max_{(\boldsymbol{f},\boldsymbol{A})}\{p(p(\boldsymbol{f},\boldsymbol{A}|\boldsymbol{g},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)\}$

  $$(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{A}})=\arg\min_{(\boldsymbol{f},\boldsymbol{A})}\left\{\|\boldsymbol{g}-\boldsymbol{A}\boldsymbol{f}\|^2+\lambda_1\|\boldsymbol{f}\|^2+\lambda_2\|\boldsymbol{A}\|^2\right\}$$
- Permutation and scale ideterminations: needs good choices for priors
- Alternate optimisation

  $$\begin{cases}\widehat{\boldsymbol{f}}=\arg\min_{\boldsymbol{f}}\left\{\|\boldsymbol{g}-\boldsymbol{A}\boldsymbol{f}\|^2+\lambda_1\|\boldsymbol{f}\|^2\right\}=(\boldsymbol{A}'\boldsymbol{A}+\lambda_1\boldsymbol{I})^{-1}\boldsymbol{A}'\boldsymbol{g}\\ \widehat{\boldsymbol{A}}=\arg\min_{\boldsymbol{A}}\left\{\|\boldsymbol{g}-\boldsymbol{A}\boldsymbol{f}\|^2+\lambda_2\|\boldsymbol{A}\|^2\right\}=\boldsymbol{g}\boldsymbol{f}'(\boldsymbol{f}\boldsymbol{f}'+\lambda_2\boldsymbol{I})\end{cases}$$
- Importance of initialization and other constraintes such as positivity

General case: Joint Estimation of $\boldsymbol{A}$ and $\boldsymbol{f}$



$$p(f_j(t)|v_{0j}) = \mathcal{N}(0, v_{0j})$$
$$p(\boldsymbol{f}(t)|\boldsymbol{v}_0) \propto \exp\left\{-\frac{1}{2}\sum_j f_j^2(t)/v_{0j}\right\}$$
$$p(\boldsymbol{A}_{ij}|\boldsymbol{A}_{0ij}, \boldsymbol{V}_{0ij}) = \mathcal{N}(\boldsymbol{A}_{0ij}, \boldsymbol{V}_{0ij})$$
$$p(\boldsymbol{A}|\boldsymbol{A}_0, \boldsymbol{V}_0) = \mathcal{N}(\boldsymbol{A}_0, \boldsymbol{V}_0)$$

$$p(\boldsymbol{g}(t)|\boldsymbol{A}, \boldsymbol{f}(t), v_\epsilon) = \mathcal{N}(\boldsymbol{A}\boldsymbol{f}(t), v_\epsilon \boldsymbol{I})$$

$$\begin{aligned} p(\boldsymbol{f}_{1..T}, \boldsymbol{A}|\boldsymbol{g}_{1..T}) &\propto p(\boldsymbol{g}_{1..T}|\boldsymbol{A}, \boldsymbol{f}_{1..T}, v_\epsilon)\, p(\boldsymbol{f}_{1..T})\, p(\boldsymbol{A}|\boldsymbol{A}_0, \boldsymbol{V}_0) \\ &\propto \prod_t p(\boldsymbol{g}(t)|\boldsymbol{A}, \boldsymbol{f}(t), v_\epsilon)\, p(\boldsymbol{f}(t)|\boldsymbol{v}_0)\, p(\boldsymbol{A}|\boldsymbol{A}_0, \boldsymbol{V}_0) \end{aligned}$$

$$p(\boldsymbol{f}(t)|\boldsymbol{g}_{1..T}, \boldsymbol{A}, v_\epsilon, \boldsymbol{v}_0) = \mathcal{N}(\widehat{\boldsymbol{f}}(t), \widehat{\boldsymbol{\Sigma}})$$

$$p(\boldsymbol{A}|\boldsymbol{g}_{1..T}, \boldsymbol{f}_{1..T}, v_\epsilon, \boldsymbol{A}_0, \boldsymbol{V}_0) = \mathcal{N}(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{V}})$$

Two approaches:

- Alternate joint MAP (JMAP) estimation
- Bayesian Variational Approximation

Joint Estimation of $\boldsymbol{A}$ and $\boldsymbol{f}$: Alternate JMAP

Let do some simplification:

$$\begin{array}{ll} \boldsymbol{v}_0 = [v_f, .., v_f]', & \text{All sources a priori same variance } v_f \\ \boldsymbol{v}_\epsilon = [v_\epsilon, .., v_\epsilon]', & \text{All noise terms a priori same variance } v_\epsilon \\ \boldsymbol{A}_0 = 0, \quad \boldsymbol{V}_0 = v_a \boldsymbol{I} & \end{array}$$

$$p(\boldsymbol{f}(t)|\boldsymbol{g}(t), \boldsymbol{A}, v_\epsilon, \boldsymbol{v}_0) = \mathcal{N}(\widehat{\boldsymbol{f}}(t), \widehat{\boldsymbol{\Sigma}})$$
$$\left\{ \begin{array}{l} \widehat{\boldsymbol{\Sigma}} = (\boldsymbol{A}'\boldsymbol{A} + \lambda_f \boldsymbol{I})^{-1} \\ \widehat{\boldsymbol{f}}(t) = (\boldsymbol{A}'\boldsymbol{A} + \lambda_f \boldsymbol{I})^{-1} \boldsymbol{A}'\boldsymbol{g}(t), \quad \lambda_f = v_\epsilon / v_f \end{array} \right.$$

$$p(\boldsymbol{A}|\boldsymbol{g}(t), \boldsymbol{f}(t), v_\epsilon, \boldsymbol{A}_0, \boldsymbol{V}_0) = \mathcal{N}(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{V}})$$
$$\left\{ \begin{array}{l} \widehat{\boldsymbol{V}} = (\boldsymbol{F}'\boldsymbol{F} + \lambda_f \boldsymbol{I})^{-1} \\ \widehat{\boldsymbol{A}} = \sum_t \boldsymbol{g}(t) \boldsymbol{f}'(t) \left(\sum_t \boldsymbol{f}(t) \boldsymbol{f}'(t) + \lambda_a \boldsymbol{I}\right)^{-1}, \quad \lambda_a = v_\epsilon / v_a \end{array} \right.$$

Joint Estimation of $\boldsymbol{A}$ and $\boldsymbol{f}$: Alternate JMAP

$$
\begin{aligned}
p(\boldsymbol{f}_{1..T}, \boldsymbol{A}|\boldsymbol{g}_{1..T}) &\propto p(\boldsymbol{g}_{1..T}|\boldsymbol{A}, \boldsymbol{f}_{1..T}, v_\epsilon)\, p(\boldsymbol{f}_{1..T})\, p(\boldsymbol{A}|\boldsymbol{A}_0, \boldsymbol{V}_0) \\
&\propto \prod_t p(\boldsymbol{g}(t)|\boldsymbol{A}, \boldsymbol{f}(t), v_\epsilon)\, p(\boldsymbol{f}(t)|\boldsymbol{z}(t))\, p(\boldsymbol{A}|\boldsymbol{A}_0, \boldsymbol{V}_0)
\end{aligned}
$$

Joint MAP: Alternate optimization

$$
\left\{
\begin{array}{ll}
\widehat{\boldsymbol{f}}(t) = (\widehat{\boldsymbol{A}}'\widehat{\boldsymbol{A}} + \lambda_f \boldsymbol{I})^{-1} \widehat{\boldsymbol{A}}'\boldsymbol{g}(t), & \lambda_f = v_\epsilon / v_f \\
\widehat{\boldsymbol{A}} = \sum_t \boldsymbol{g}(t)\widehat{\boldsymbol{f}}'(t) \left(\sum_t \widehat{\boldsymbol{f}}(t)\widehat{\boldsymbol{f}}'(t) + \lambda_a \boldsymbol{I}\right)^{-1} & \lambda_a = v_\epsilon / v_a
\end{array}
\right.
$$

Alternate optimization Algorithm:

$$
\begin{array}{ccccc}
\boldsymbol{A}^{(0)} \longrightarrow \widehat{\boldsymbol{A}} \longrightarrow & \boxed{\left(\widehat{\boldsymbol{A}}'\widehat{\boldsymbol{A}} + \lambda_f \boldsymbol{I}\right)^{-1} \widehat{\boldsymbol{A}}'\boldsymbol{g}} & \longrightarrow \widehat{\boldsymbol{f}}(t) \\
\uparrow & & \downarrow \\
\widehat{\boldsymbol{A}} \longleftarrow & \boxed{\sum_t \boldsymbol{g}(t)\widehat{\boldsymbol{f}}'(t) \left(\sum_t \widehat{\boldsymbol{f}}(t)\widehat{\boldsymbol{f}}'(t) + \lambda_a \boldsymbol{I}\right)^{-1}} & \longleftarrow \widehat{\boldsymbol{f}}(t)
\end{array}
$$

Variational Bayesian Approximation

Can we do better? Yes, VBA is a good solution.

- Main idea: Approximate a joint pdf $p(\boldsymbol{x})$ difficult to handle by a simpler one (for example a separable one $q(\boldsymbol{x}) = \prod_j q_j(x_j)$)
- Criterion: minimize

$$\mathsf{KL}(q|p) = \int q \ln \frac{q}{p} = \left\langle \ln \frac{q}{p} \right\rangle_q = \sum_j H(q_j) - < \ln p(\boldsymbol{x}) >_q$$

- Solution: $q_j(x_j) \propto \exp\left\{- < \ln p(\boldsymbol{x}) >_{q_{-j}}\right\}$
- In our case: Approximate $p(\boldsymbol{f}, \boldsymbol{A}|\boldsymbol{g})$ by a separable one $q(\boldsymbol{f}, \boldsymbol{A}) = q_1(\boldsymbol{f})q_2(\boldsymbol{A})$
- Solution obtained by alternate optimization:

$$\left\{ \begin{array}{l} q_1(\boldsymbol{f}) \propto \exp\left\{- < \ln p(\boldsymbol{f}, \boldsymbol{A}|\boldsymbol{g}) >_{q_2(\boldsymbol{A})}\right\} \\ q_2(\boldsymbol{A}) \propto \exp\left\{- < \ln p(\boldsymbol{f}, \boldsymbol{A}|\boldsymbol{g}) >_{q_1(\boldsymbol{f})}\right\} \end{array} \right.$$

Joint Estimation: Variational Bayesian Approximation

$$p(\boldsymbol{f}_{1..T}, \boldsymbol{A}|\boldsymbol{g}_{1..T}) \longrightarrow q_1(\boldsymbol{f}_{1..T}|\widetilde{\boldsymbol{A}}, \boldsymbol{g}_{1..T})\; q_2(\boldsymbol{A}|\widetilde{\boldsymbol{f}}_{1..T}, \boldsymbol{g}_{1..T})$$

$$q_1(\boldsymbol{f}(t)|\boldsymbol{g}(t), \boldsymbol{A}, v_\epsilon, \boldsymbol{v}_0) = \mathcal{N}(\widehat{\boldsymbol{f}}(t), \widehat{\boldsymbol{\Sigma}})$$
$$\left\{\begin{array}{l} \widehat{\boldsymbol{\Sigma}} = (\boldsymbol{A}'\boldsymbol{A} + \lambda_f \widehat{\boldsymbol{V}})^{-1} \\ \widehat{\boldsymbol{f}}(t) = (\boldsymbol{A}'\boldsymbol{A} + \lambda_f \widehat{\boldsymbol{V}})^{-1} \boldsymbol{A}'\boldsymbol{g}(t), \quad \lambda_f = v_\epsilon/v_f \end{array}\right.$$

$$q_2(\boldsymbol{A}|\boldsymbol{g}(t), \boldsymbol{f}(t), v_\epsilon, \boldsymbol{A}_0, \boldsymbol{V}_0) = \mathcal{N}(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{V}})$$
$$\left\{\begin{array}{l} \widehat{\boldsymbol{V}} = (\boldsymbol{F}'\boldsymbol{F} + \lambda_f \widehat{\boldsymbol{\Sigma}})^{-1} \\ \widehat{\boldsymbol{A}} = \sum_t \boldsymbol{g}(t)\boldsymbol{f}'(t) \left(\sum_t \boldsymbol{f}(t)\boldsymbol{f}'(t) + \lambda_a \widehat{\boldsymbol{\Sigma}}\right)^{-1}, \quad \lambda_a = v_\epsilon/v_a \end{array}\right.$$

$\boldsymbol{A}^{(0)} \longrightarrow \widehat{\boldsymbol{A}} \longrightarrow$, $\boldsymbol{V}^{(0)} \longrightarrow \widehat{\boldsymbol{V}} \longrightarrow$

$$\boxed{\begin{array}{l} \widehat{\boldsymbol{f}}(t) = \left(\widehat{\boldsymbol{A}}'\widehat{\boldsymbol{A}} + \lambda_f \widehat{\boldsymbol{V}}\right)^{-1} \widehat{\boldsymbol{A}}'\boldsymbol{g}(t) \\ \widehat{\boldsymbol{\Sigma}} = (\boldsymbol{A}'\boldsymbol{A} + \lambda_f \widehat{\boldsymbol{V}})^{-1} \end{array}}$$

$\longrightarrow \widehat{\boldsymbol{f}}(t)$, $\longrightarrow \widehat{\boldsymbol{\Sigma}}$

$\Uparrow$ $\qquad\qquad$ $\Downarrow$

$\widehat{\boldsymbol{A}} \longleftarrow$, $\widehat{\boldsymbol{V}} \longleftarrow$

$$\boxed{\begin{array}{l} \widehat{\boldsymbol{A}} = \sum_t \boldsymbol{g}(t)\widehat{\boldsymbol{f}}'(t) \left(\sum_t \widehat{\boldsymbol{f}}(t)\widehat{\boldsymbol{f}}'(t) + \lambda_a \widehat{\boldsymbol{\Sigma}}\right)^{-1} \\ \widehat{\boldsymbol{V}} = (\boldsymbol{F}'\boldsymbol{F} + \lambda_f \widehat{\boldsymbol{\Sigma}})^{-1} \end{array}}$$

$\longleftarrow \widehat{\boldsymbol{f}}(t)$, $\longleftarrow \widehat{\boldsymbol{\Sigma}}$

Bayesian Sparse Sources Separation

Three main steps:

- Assigning priors (sparsity enforcing):
  - Simple priors: $p(\boldsymbol{f})$ and $p(\boldsymbol{A})$
  - Hierarchical priors: $p(\boldsymbol{f}|\boldsymbol{z})\,p(\boldsymbol{z})$ and $p(\boldsymbol{A}|\boldsymbol{q})\,p(\boldsymbol{q})$
- Obtaining the expressions of $p(\boldsymbol{f},\boldsymbol{A},\boldsymbol{\theta}|\boldsymbol{g})$ or $p(\boldsymbol{f},\boldsymbol{A},\boldsymbol{z},\boldsymbol{q},\boldsymbol{\theta}|\boldsymbol{g})$
- Doing the computations:
  - Joint optimization of $p(\boldsymbol{f},\boldsymbol{A},\boldsymbol{\theta}|\boldsymbol{g})$;
  - MCMC Gibbs sampling methods which need generation of samples from the conditionals $p(\boldsymbol{f}|\boldsymbol{A},\boldsymbol{\theta},\boldsymbol{g})$, $p(\boldsymbol{A}|\boldsymbol{f},\boldsymbol{\theta},\boldsymbol{g})$ and $p(\boldsymbol{\theta}|\boldsymbol{f},\boldsymbol{A},\boldsymbol{g})$;
  - Bayesian Variational Approximation (BVA) methods which approximate $p(\boldsymbol{f},\boldsymbol{A},\boldsymbol{\theta}|\boldsymbol{g})$ by a separable one

$$q(\boldsymbol{f},\boldsymbol{A},\boldsymbol{\theta}|\boldsymbol{g}) = q_1(\boldsymbol{f}|\widetilde{\boldsymbol{A}},\widetilde{\boldsymbol{\theta}},\boldsymbol{g})\; q_2(\boldsymbol{A}|\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{\theta}},\boldsymbol{g})\; q_3(\boldsymbol{\theta}|\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{A}},\boldsymbol{g})$$

and then using them for the estimation.

Conclusions

- General source separation problem
  - Estimation of $\boldsymbol{f}$ when $\boldsymbol{A}$ is known
  - Estimation of $\boldsymbol{A}$ when the sources $\boldsymbol{f}$ are known
  - Joint estimation of the sources $\boldsymbol{f}$ and the mixing matrix $\boldsymbol{A}$
- Priors which enforce sparsity:
  - Generalized Gaussian, Student-t, Elastic nets, ...
  - Scaled Gaussian Mixture, Mixture of Gaussians or Gammas, Bernoulli-Gaussian
- Computational tools:
  - Alternate optimization of JMAP criterion
  - MCMC
  - Variational Bayesian Approximation
- Advanced Bayesian methods: Non-Gaussian, Dependent and nonstationnary signals and images.
- Some domaines of applications
  - Acoustic Source localization, Radar and SAR imaging, Spectrometry, Cosmic Microwave Background, Sattelite Image separation, Hyperspectral image processing

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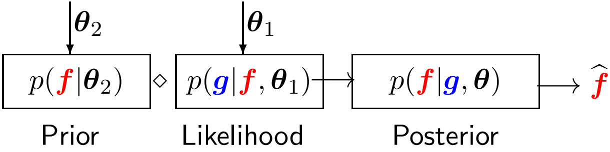
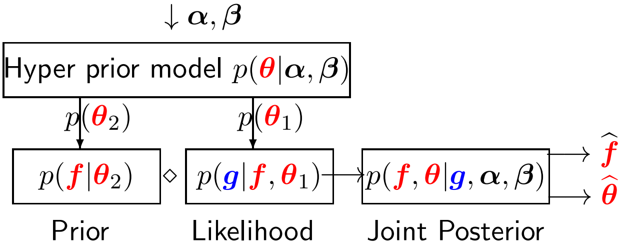
Summary of Bayesian estimation with different levels

- Simple Bayesian Model and Estimation



- Full Bayesian Model and Hyperparameter Estimation scheme



Summary of Bayesian estimation with different levels

- Marginalization for Hyperparameter Estimation

$$\boxed{p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g})} \longrightarrow \boxed{p(\boldsymbol{\theta}|\boldsymbol{g})} \longrightarrow \widehat{\boldsymbol{\theta}} \longrightarrow \boxed{p(\boldsymbol{f}|\widehat{\boldsymbol{\theta}},\boldsymbol{g})} \longrightarrow \widehat{\boldsymbol{f}}$$

Joint Posterior　　Marginalize over $\boldsymbol{f}$

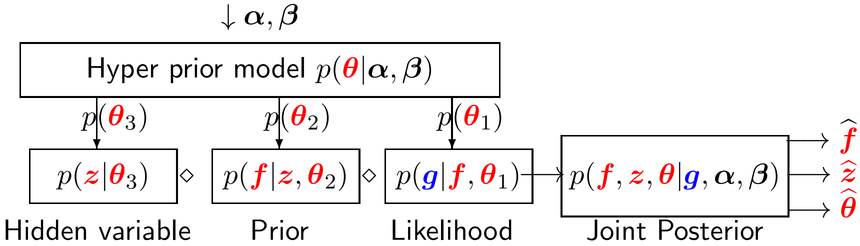
- Full Bayesian Model with a Hierarchical Prior Model

$$\begin{array}{ccccccc} \boldsymbol{\theta}_3 & & \boldsymbol{\theta}_2 & & \boldsymbol{\theta}_1 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \boxed{p(\boldsymbol{z}|\boldsymbol{\theta}_3)} & \diamond & \boxed{p(\boldsymbol{f}|\boldsymbol{z},\boldsymbol{\theta}_2)} & \diamond & \boxed{p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1)} & \longrightarrow & \boxed{p(\boldsymbol{f},\boldsymbol{z}|\boldsymbol{g},\boldsymbol{\theta})} \longrightarrow \widehat{\boldsymbol{f}},\ \widehat{\boldsymbol{z}} \end{array}$$

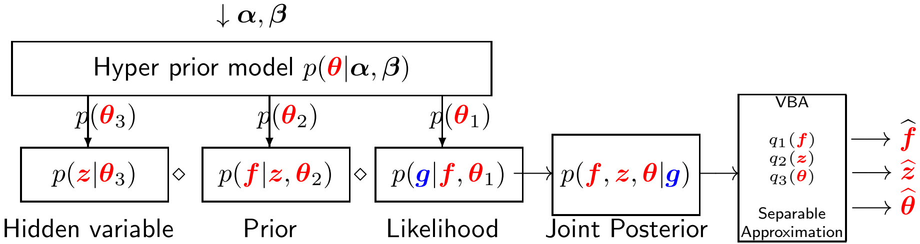
Hidden variable　　Prior　　Likelihood　　Joint Posterior

Summary of Bayesian estimation with different levels

- Full Bayesian Hierarchical Model with Hyperparameter Estimation



- Full Bayesian Hierarchical Model and Variational Approximation



Prior models with hidden variables

- Example 1: MoG model:

$$p(f_j|\lambda, v_1, v_0) = \lambda \mathcal{N}(f_j|0, v_1) + (1-\lambda)\mathcal{N}(f_j|0, v_0)$$

$$\left\{ \begin{array}{l} p(f_j|z_j = 0, v_0) = \mathcal{N}(f_j|0, v_0), \\ p(f_j|z_j = 1, v_1) = \mathcal{N}(f_j|0, v_1), \end{array} \right. \text{ and } \left\{ \begin{array}{l} P(z_j = 0) = \lambda, \\ P(z_j = 1) = 1 - \lambda \end{array} \right.$$

$$\left\{ \begin{array}{l} p(\boldsymbol{f}|\boldsymbol{z}) = \prod_j p(f_j|z_j) = \prod_j \mathcal{N}\left(f_j|0, v_{z_j}\right) \propto \exp\left\{-\frac{1}{2}\sum_j \frac{f_j^2}{v_{z_j}}\right\} \\ P(z_j = 1) = \lambda, \qquad P(z_j = 0) = 1 - \lambda \end{array} \right.$$

- Example 2: Student-t model

$$\mathcal{S}t(f|\nu) = \int_0^\infty \mathcal{N}(f|, 0, 1/z)\, \mathcal{G}(z|\alpha, \beta)\; \mathrm{d}z, \quad \text{with } \alpha = \beta = \nu/2$$

$$\left\{ \begin{array}{ll} p(\boldsymbol{f}|\boldsymbol{z}) & = \prod_j p(f_j|z_j) = \prod_j \mathcal{N}(f_j|0, 1/z_j) \propto \exp\left\{-\frac{1}{2}\sum_j z_j f_j^2\right\} \\ p(\boldsymbol{z}|\alpha, \beta) & = \prod_j \mathcal{G}(z_j|\alpha, \beta) \propto \prod_j z_j^{(\alpha-1)} \exp\{-\beta z_j\} \\ & \propto \exp\left\{\sum_j (\alpha-1)\ln z_j - \beta z_j\right\} \\ p(\boldsymbol{f}, \boldsymbol{z}|\alpha, \beta) & \propto \exp\left\{-\frac{1}{2}\sum_j z_j f_j^2 + (\alpha-1)\ln z_j - \beta z_j\right\} \end{array} \right.$$

- With these models we have:

Bayesian Computation and Algorithms

- Often, the expression of $p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g})$ is complex.
- Its optimization (for Joint MAP) or
  its marginalization or integration (for Marginal MAP or PM)
  is not easy
- Two main techniques:
  MCMC and Variational Bayesian Approximation (VBA)
- MCMC:
  Needs the expressions of the conditionals
  $p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta}, \boldsymbol{g})$, $p(\boldsymbol{z}|\boldsymbol{f}, \boldsymbol{\theta}, \boldsymbol{g})$, and $p(\boldsymbol{\theta}|\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{g})$
- VBA: Approximate $p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g})$ by a separable one

$$q(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) = q_1(\boldsymbol{f})\, q_2(\boldsymbol{z})\, q_3(\boldsymbol{\theta})$$

  and do any computations with these separable ones.

General case

$$\boldsymbol{g}(t) = \boldsymbol{A}\boldsymbol{f}(t) + \boldsymbol{\epsilon}(t), \quad t \in [1, \cdots, T]$$

$$p(\boldsymbol{g}(t)|\boldsymbol{f}(t), \boldsymbol{A}, v_\epsilon, \boldsymbol{v}_0) = \mathcal{N}(\boldsymbol{g}(t) - \boldsymbol{A}\boldsymbol{f}(t), v_\epsilon \boldsymbol{I}) p(\boldsymbol{f}(t)|v_f) = \mathcal{N}(0, v_0 \boldsymbol{I})$$

$$p(\boldsymbol{f}(t)|\boldsymbol{g}(t), \boldsymbol{A}, v_\epsilon, \boldsymbol{v}_0) = \mathcal{N}(\widehat{\boldsymbol{f}}(t), \widehat{\boldsymbol{\Sigma}})$$

Full Bayesian and Variational Bayesian Approximation

- Full Bayesian: $p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)\, p(\boldsymbol{\theta})$
- Approximate $p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g})$ by $q(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}) = q_1(\boldsymbol{f}|\boldsymbol{g})\, q_2(\boldsymbol{\theta}|\boldsymbol{g})$ and then continue computations.
- Criterion $\text{KL}(q(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}) : p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g}))$
- $\text{KL}(q : p) = \int\int q \ln q/p = \int\int q_1 q_2 \ln \frac{q_1 q_2}{p}$
- Iterative algorithm $q_1 \longrightarrow q_2 \longrightarrow q_1 \longrightarrow q_2, \cdots$

$$\left\{ \begin{array}{ll} \widehat{q}_1(\boldsymbol{f}) & \propto \exp\left\{ \langle \ln p(\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{\theta}; \mathcal{M}) \rangle_{\widehat{q}_2(\boldsymbol{\theta})} \right\} \\ \widehat{q}_2(\boldsymbol{\theta}) & \propto \exp\left\{ \langle \ln p(\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{\theta}; \mathcal{M}) \rangle_{\widehat{q}_1(\boldsymbol{f})} \right\} \end{array} \right.$$

$\boxed{p(\boldsymbol{f}, \boldsymbol{\theta}|\boldsymbol{g})} \longrightarrow$ **[Variational Bayesian Approximation]** $\longrightarrow q_1(\boldsymbol{f}) \longrightarrow \widehat{\boldsymbol{f}}$, $\longrightarrow q_2(\widehat{\boldsymbol{\theta}}) \longrightarrow \widehat{\boldsymbol{\theta}}$

Bayesian estimation approach

$$\boldsymbol{x}(t) = \boldsymbol{A}\,\boldsymbol{s}(t) + \boldsymbol{\epsilon}(t) \quad \text{ou} \quad \boldsymbol{x}_{1..T} = \boldsymbol{A}\,\boldsymbol{s}_{1..T} + \boldsymbol{\epsilon}_{1..T} \quad \text{ou} \quad \boldsymbol{X} = \boldsymbol{A}\boldsymbol{S} + \boldsymbol{E}$$

$$p(\boldsymbol{A}, \boldsymbol{s}_{1..T} | \boldsymbol{x}_{1..T}) \propto p(\boldsymbol{x}_{1..T} | \boldsymbol{A}, \boldsymbol{s}_{1..T})\, p(\boldsymbol{A})\, p(\boldsymbol{s}_{1..T})$$

$$\boldsymbol{x}(\boldsymbol{r}) = \boldsymbol{A}\,\boldsymbol{s}(\boldsymbol{r}) + \boldsymbol{\epsilon}(\boldsymbol{r}) \quad \text{ou} \quad \boldsymbol{x}(\boldsymbol{r}) = \boldsymbol{A}\,\boldsymbol{s}_{\boldsymbol{r}\in\mathcal{R}} + \boldsymbol{\epsilon}(\boldsymbol{r}) \quad \text{ou} \quad \boldsymbol{X} = \boldsymbol{A}\boldsymbol{S} + \boldsymbol{E}$$

$$p(\boldsymbol{A}, \boldsymbol{s}_{\boldsymbol{r}\in\mathcal{R}} | \boldsymbol{x}(\boldsymbol{r})) \propto p(\boldsymbol{x}(\boldsymbol{r}) | \boldsymbol{A}, \boldsymbol{s}_{\boldsymbol{r}\in\mathcal{R}})\, p(\boldsymbol{A})\, p(\boldsymbol{s}_{\boldsymbol{r}\in\mathcal{R}})$$

$$p(\boldsymbol{A}, \boldsymbol{S} | \boldsymbol{X}) \propto p(\boldsymbol{X} | \boldsymbol{A}, \boldsymbol{S})\, p(\boldsymbol{A})\, p(\boldsymbol{S})$$

$$p(\boldsymbol{X} | \boldsymbol{A}, \boldsymbol{S}) = \mathcal{N}(\boldsymbol{A}\boldsymbol{S}, \boldsymbol{\Sigma}_\epsilon), \quad p(\boldsymbol{A}) = \mathcal{N}(\boldsymbol{A}_0, \boldsymbol{\Sigma}_0) \text{ or uniform}$$

◊ Important step: Choice of $p(\boldsymbol{S})$

Bayesian estimation approach

$$\boldsymbol{x}(t) = \boldsymbol{A}\,\boldsymbol{s}(t) + \boldsymbol{\epsilon}(t)$$

$$p(\boldsymbol{A}, \boldsymbol{s}_{1..T}|\boldsymbol{x}_{1..T}) \propto p(\boldsymbol{x}_{1..T}|\boldsymbol{A}, \boldsymbol{s}_{1..T})\, p(\boldsymbol{A})\, p(\boldsymbol{s}_{1..T})$$

◊ 3 directions:

1. Joint estimation: $(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{s}}_{1..T})$ using $p(\boldsymbol{A}, \boldsymbol{s}_{1..T}|\boldsymbol{x}_{1..T})$. For example JMAP:

$$(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{s}}_{1..T}) = \arg \max_{(\boldsymbol{A}, \boldsymbol{s}_{1..T})} \{J(\boldsymbol{A}, \boldsymbol{s}_{1..T}) = \ln\, p(\boldsymbol{A}, \boldsymbol{s}_{1..T}|\boldsymbol{x}_{1..T})\}$$

2. $\boldsymbol{A}$ estimation: $\widehat{\boldsymbol{A}}$ using $p(\boldsymbol{A}|\boldsymbol{x}_{1..T})$. For example:

$$\widehat{\boldsymbol{A}} = \arg\max_{\boldsymbol{A}} \{J(\boldsymbol{A}) = \ln\, p(\boldsymbol{A}|\boldsymbol{x}_{1..T})\}$$

3. $\boldsymbol{s}$ estimation: $\widehat{\boldsymbol{s}}$ using $p(\boldsymbol{s}_{1..T}|\boldsymbol{x}_{1..T})$. For example:

$$\widehat{\boldsymbol{s}}_{1..T} = \arg\max_{\boldsymbol{s}} \{J(\boldsymbol{s}_{1..T}) = \ln\, p(\boldsymbol{s}_{1..T}|\boldsymbol{x}_{1..T})\}$$

Gaussian white case: PCA, MNF, PMF, NMF and SOBI

- White and Gaussian signals $\boldsymbol{s}(t), \boldsymbol{\epsilon}(t) \longrightarrow \boldsymbol{x}(t)$:

$$\boldsymbol{x}(t) = \boldsymbol{A}\,\boldsymbol{s}(t) + \boldsymbol{\epsilon}(t) \longrightarrow \boldsymbol{x} = \boldsymbol{A}\,\boldsymbol{s} + \boldsymbol{\epsilon}$$

$$p(\boldsymbol{x}, \boldsymbol{s}|\boldsymbol{A}) = p(\boldsymbol{x}|\boldsymbol{A}, \boldsymbol{s})\,p(\boldsymbol{s})$$

$$p(\boldsymbol{x}|\boldsymbol{A}, \boldsymbol{s}) = \mathcal{N}(\boldsymbol{A}\boldsymbol{s}, \boldsymbol{\Sigma}_\epsilon), \quad p(\boldsymbol{s}) = \mathcal{N}(0, \boldsymbol{\Sigma}_s) \longrightarrow p(\boldsymbol{x}|\boldsymbol{A}) = \mathcal{N}(0, \boldsymbol{A}\boldsymbol{\Sigma}$$

- PCA : Estimate $\boldsymbol{\Sigma}_x$ by $\frac{1}{T}\sum_t \boldsymbol{x}(t)\boldsymbol{x}'(t)$,
  svd and keep all the non-zero svd: $\boldsymbol{\Sigma}_x = \boldsymbol{A}\boldsymbol{\Sigma}_s\boldsymbol{A}'$
- Minimum Norm Factorization (MNF) :
  Estimate $\boldsymbol{\Sigma}_x$, svd and keep all svd $\geq \sigma_\epsilon$: $\boldsymbol{\Sigma}_x = \boldsymbol{A}\boldsymbol{\Sigma}_s\boldsymbol{A}' + \boldsymbol{\Sigma}_\epsilon$
- Positive Matrix Factorization (MNF) :
  Decompose $\boldsymbol{\Sigma}_x$ in positive definite matrices [Paatero & Tapper, 94]
- Non-negative Matrix Factorization (NMF) :
  Decompose $\boldsymbol{\Sigma}_x$ in Non-negative definite matrices [Lee& Seung,99]

Gaussian white case: PCA, MNF, PMF, NMF and SOBI (2)

Accounting for non stationnarity

◊ SOBI : $1..T = 1..T_1..T_2..T_k..T_{k+1}..T$

$$\boldsymbol{\Sigma}_x(k) = \frac{1}{T_{k+1} - T_k} \sum_{t=T_k}^{T_{k+1}-1} \boldsymbol{x}(t)\boldsymbol{x}'(t)$$

Joint Diagonalization of $\boldsymbol{\Sigma}_x(k)$

Non Gaussian white case: ICA, JADE, NNICA and Tappered ICA

◊ White Non Gaussian signals and Exact model (no noise):

$$\boldsymbol{s}(t) \longrightarrow \boldsymbol{x}(t) \longrightarrow \boldsymbol{y}(t) = \boldsymbol{A}^{-1}\boldsymbol{x}(t) \longrightarrow \boldsymbol{y}(t) = \boldsymbol{B}\boldsymbol{x}(t)$$

ICA: Find $\boldsymbol{B}$ in such a way that the components of $\boldsymbol{y}$ be the most independent

Different measures of independencies:

$$S(\boldsymbol{y}) = -\int p(y_i) \ln p(y_i) \, \mathrm{d}y_i$$

$$KL(p(\boldsymbol{y}) : \prod_i p(y_i)) = \int p(y_i) \ln \frac{\prod_i p(y_i)}{p(\boldsymbol{y}} \, \mathrm{d}y_i$$

Different choices and approximations for $p(y_i) \longrightarrow$ contrast functions, cumulants basis criteria

Non Gaussian white case: ICA, JADE, NNICA and Tappered ICA (2)

◊ White Non Gaussian signals (Accounting for noise)

$$\boldsymbol{x}(t) = \boldsymbol{A}\,\boldsymbol{s}(t) + \boldsymbol{\epsilon}(t) \longrightarrow \boldsymbol{x} = \boldsymbol{A}\,\boldsymbol{s} + \boldsymbol{\epsilon}$$

$$p(\boldsymbol{x}|\boldsymbol{A}, \boldsymbol{\Sigma}_\epsilon) = \int p(\boldsymbol{x}|\boldsymbol{A}, \boldsymbol{s}, \boldsymbol{\Sigma}_\epsilon)\, p(\boldsymbol{s})\, \mathsf{d}\boldsymbol{s}$$

◊ ICA (Maximum Likelihood) :

$$\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{\Sigma}_\epsilon}) = \arg\max_{\boldsymbol{\theta}} \{p(\boldsymbol{x}|\boldsymbol{A}, \boldsymbol{\theta})\}$$

◊ EM iterative algorithm :

$$\begin{aligned} Q(\boldsymbol{\theta}, \boldsymbol{\theta}') &= \mathsf{E}\left\{\ln p(\boldsymbol{x}, \boldsymbol{s}|\boldsymbol{\theta})|\boldsymbol{\theta}'\right\} \\ \boldsymbol{\theta}' &= \arg\max_{\boldsymbol{\theta}} \left\{Q(\boldsymbol{\theta}, \boldsymbol{\theta}')\right\} \end{aligned}$$

Choice of a priori and JMAP algorithm

◊ Sources $\boldsymbol{s} \longrightarrow$ Mixture of Gaussians [Moulines97]:

$$p(s_j) = \sum_{i=1}^{q_j} \alpha_{ji} \mathcal{N}\left(m_{ji}, \sigma_{ji}^2\right), \qquad j = 1..n$$

◊ Mixing matrix $\boldsymbol{A} \longrightarrow$ a priori Gaussian for its elements:

$$p(\boldsymbol{A}_{ij}) = \mathcal{N}\left(\mu_{ij}, \sigma_{a,ij}^2\right)$$

◊ Scalar iteratif algorithms:

$$\left\{ \begin{array}{rcl} \widehat{\boldsymbol{s}}_j(t)^{(k+1)} & = & \arg\max_{\boldsymbol{s}_j} \left\{ \ln p\left(\boldsymbol{s}_j | \boldsymbol{x}(t), \widehat{\boldsymbol{A}}^{(k)}, \widehat{\boldsymbol{s}}_{l \neq j}(t)^{(k)}\right) \right\} \\ \widehat{\boldsymbol{A}}^{(k+1)} & = & \arg\max_{\boldsymbol{A}} \left\{ \ln p\left(\boldsymbol{A} | \widehat{\boldsymbol{s}}_{1..T}^{(k+1)}, \boldsymbol{x}_{1..T}\right) \right\} \end{array} \right.$$