Inverse problems arising in different synthetic aperture radar imaging systems and a general Bayesian approach for them

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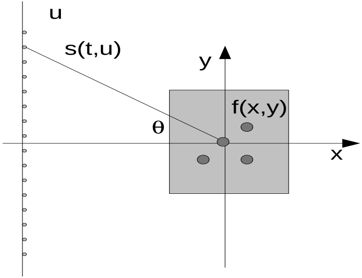
SPIE, Electronic Imaging, San Francisco, January 23-27, 2011

Summary

- Introduction to SAR imaging
- Monostatic, Bistatic and Multistatic SAR imaging
- Forward modeling as a Fourier Synthesis inverse problem
- Classical inversion methods
  - Inverse Fourier Transform
  - Least square and deterministic regularization
- Bayesian estimation approach
- Proposed method of joint data fusion and super-resolution reconstruction
- Simulation and experimental data results
- Conclusions and Discussions

Synthetic Aperture Radar (SAR) imaging



$$s(t,u) = \sum_m \sum_n f(m,n)\, p(t - \tau_{m,n}(u))$$

$$s(t,u) = \iint f(x,y)\, p(t - \tau(x,y,u))\, \mathrm{d}x\, \mathrm{d}y$$

$$s(\omega, \theta(u)) = P(\omega) \iint f(x,y)\, \exp\{-j\omega\tau(x,y,\theta(u))\}\, \mathrm{d}x\, \mathrm{d}y$$

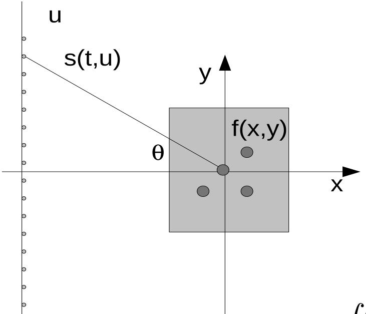
$$\boldsymbol{k} = \begin{bmatrix} k_x \\ k_y \end{bmatrix} = \begin{bmatrix} k\cos(\theta) \\ k\sin(\theta) \end{bmatrix} \qquad |\boldsymbol{k}| = k = \omega/c$$

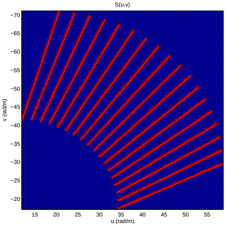
Monostatic, Bistatic and Multstatic cases

- Mono-static case (same transmitter-receivers)

$$s(\omega,\theta(u)) = P(\omega)\iint f(x,y)\,\exp\left\{-j\omega\tau(x,y,\theta(u))\right\}\,\mathrm{d}x\,\mathrm{d}y$$

$$\tau(x,y,u(\theta)) = \frac{2}{c}\sqrt{x^2+(y-u)^2} = \frac{2}{\omega}(k_x x + k_y(y-u))$$





$$k_x = k\,\cos(\theta)$$

$$k_y = k\,\sin(\theta)$$

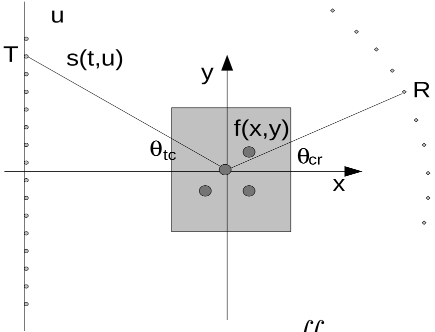
$$\begin{aligned} s(\omega,\theta(u)) &= P(\omega)\iint f(x,y)\,\exp\left\{-j\omega\tau(x,y,\theta(u))\right\}\,\mathrm{d}x\,\mathrm{d}y \\ &= P(\omega)\iint f(x,y)\,\exp\left\{-j(k_x x + k_y y)\right\}\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

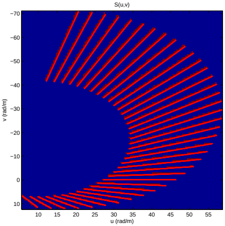
Bistatic and Multstatic cases

- Bistatic case (one transmitter, many receivers)
- Multistatic case (one transmitter, many receivers)

$$s(t, u(\theta)) = \iint f(x, y)\, p(t - \tau_{tc}(x, y) - \tau_{rc}(x, y, u(\theta)))\, \mathrm{d}x\, \mathrm{d}y$$

$$\tau_{tc} + \tau_{cr} = \frac{2}{\omega}(k_x x + k_y (y - u))$$





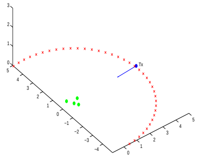
$$k_x = k\,(\cos(\theta_{tc}) + \cos(\theta_{cr}))$$
$$k_y = k\,(\sin(\theta_{tc}) + \sin(\theta_{cr}))$$

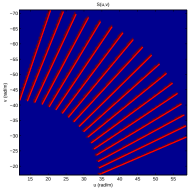
$$s(\omega, \theta(u)) = P(\omega) \iint f(x, y)\, \exp\{-j(k_x x + k_y y)\}\, \mathrm{d}x\, \mathrm{d}y$$

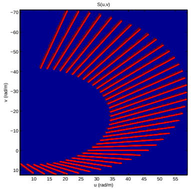
Monostatic, Bistatic and Multstatic cases

$$s(\omega,\theta(u)) = P(\omega) \iint f(x,y)\, \exp\{-j(k_x x + k_y y)\}\, \mathrm{d}x\, \mathrm{d}y$$

$$|P(\omega)| = 1 \quad \omega \in [\omega_{min}, \omega_{max}]$$







T/T/R config. | Monostatic | Bistatic & Multistatic

Monostatic:

$$\begin{cases} k_x = k\, \cos(\theta) \\ k_y = k\, \sin(\theta) \end{cases}$$

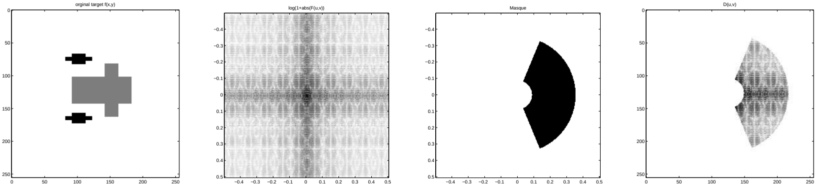
Bistatic & Multistatic:

$$\begin{cases} k_x = k\,(\cos(\theta_{tc}) + \cos(\theta_{cr})) \\ k_y = k\,(\sin(\theta_{tc}) + \sin(\theta_{cr})) \end{cases}$$

- For each position of transmitter/receiver we get information on the Fourier domain of the scene on a ligne segment which length is proportional to the bandwidth of the transmitted signal and its orientation depends on the relative positions of Transmitter-Scene-Receiver

Forward modeling as a Fourier Synthesis inverse problem

$$G(k_x, k_y) = M(k_x, k_y) F(k_x, k_y)$$



$f(x, y)$ $F(k_x, k_y)$ $M(k_x, k_y)$ $G(k_x, k_y)$

Forward model:

$$f(x, y) \longrightarrow G(k_x, k_y), M(k_x, k_y)$$

Inverse problem:

$$G(k_x, k_y), M(k_x, k_y) \longrightarrow f(x, y)$$

Fourier synthesis inverse problem

$$g(u_i, v_i) = \int\int f(x,y) \exp\{-j(u_i x + v_i y)\} \,\mathrm{d}x\mathrm{d}y$$

$$g = \mathcal{H} f + \epsilon$$

$$g(\boldsymbol{s}_i) = \int f(\boldsymbol{r})\, h(\boldsymbol{s}_i, \boldsymbol{r})\, \mathrm{d}\boldsymbol{r} \longrightarrow \text{Discretization: } f(\boldsymbol{r}) = \sum_j f_j b_j(\boldsymbol{r})$$

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- $\boldsymbol{H}$ Forward FT matrix
- $\boldsymbol{H}\boldsymbol{f}$: Fourier transform of f
- $\boldsymbol{H}^t\boldsymbol{g}$: IFT of g assuming the missing data are equal to zero.
- Remark: $\boldsymbol{H}\boldsymbol{H}^t = \boldsymbol{I}$ but $\boldsymbol{H}^t\boldsymbol{H} \neq \boldsymbol{I}$.

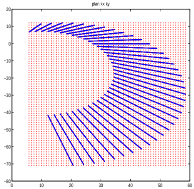
Classical analytical methods

$$
\begin{aligned}
s(\omega,\theta(u)) &= P(\omega)\iint f(x,y)\exp\{-j\omega\tau(x,y,\theta(u))\}\ \mathrm{d}x\ \mathrm{d}y \\
&= P(\omega)\iint f(x,y)\exp\{-j(k_x x + k_y y)\}\ \mathrm{d}x\ \mathrm{d}y
\end{aligned}
$$

Assuming $|P(\omega)| = 1$, we can write

$$
f(x,y) = \iint s(\omega,\theta(u))\exp\{+j(k_x x + k_y y)\}\ \mathrm{d}k_x\ \mathrm{d}k_y
$$

- Interpolation: $s(\omega,\theta(u)) \longrightarrow F(k_x,k_y)$



- Inverse Fourier Transform: $F(k_x,k_y) \longrightarrow IFT \longrightarrow f(x,y)$
- All the unknown values of $F(k_x,k_y)$ are assumed to be equal to zero.

Bayesian estimation approach

$$\mathcal{M}: \quad \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Observation model $\mathcal{M}$ + Hypothesis on the noise $\boldsymbol{\epsilon} \longrightarrow$
$$p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M}) = p_{\boldsymbol{\epsilon}}(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f})$$
- A priori information $\quad p(\boldsymbol{f}|\mathcal{M})$
- Bayes : $$p(\boldsymbol{f}|\boldsymbol{g};\mathcal{M}) = \frac{p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M})\, p(\boldsymbol{f}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$$

Link with regularization :

Maximum A Posteriori (MAP) :

$$\begin{aligned}\widehat{\boldsymbol{f}} &= \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{g}|\boldsymbol{f})\; p(\boldsymbol{f})\} \\ &= \arg\min_{\boldsymbol{f}} \{-\ln p(\boldsymbol{g}|\boldsymbol{f}) - \ln p(\boldsymbol{f})\}\end{aligned}$$

with $\quad Q(\boldsymbol{g}, \boldsymbol{H}\boldsymbol{f}) = -\ln p(\boldsymbol{g}|\boldsymbol{f}) \quad$ and $\quad \lambda\Omega(\boldsymbol{f}) = -\ln p(\boldsymbol{f})$

But, Bayesian inference is not only limited to MAP

Proposed method: Bayesian with different a priori

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \left\{ J(\boldsymbol{f}) = -\ln p(\boldsymbol{g}|\boldsymbol{f}) - \ln p(\boldsymbol{f}) \right\}$$

$$p(\boldsymbol{g}|\boldsymbol{f}) \propto \exp\left\{ \frac{1}{2\sigma_\epsilon^2} \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 \right\}$$

- Generalized Gaussian $p(\boldsymbol{f}) \propto \exp\left\{ \gamma \sum_j |f_j|^2 \right\}$
- Cauchy $p(\boldsymbol{f}) \propto \exp\left\{ \gamma \sum_j \ln(1 - |f_j|^2 \right\}$
- Sparse Gaussian $p(\boldsymbol{f}) \propto \exp\left\{ \sum_j \gamma_j |f_j|^2 \right\}$
- Generalized Gauss-Markov

$$p(\boldsymbol{f}) \propto \exp\left\{ \gamma \sum_j |f_j - f_{j-1}|^\beta \right\} \propto \exp\left\{ \gamma \sum_j |[\boldsymbol{D}\boldsymbol{f}]_j|^\beta \right\}$$

- Gauss-Markov-Potts model

Sparse Gaussian prior model

- Sparse Gaussian model:

$$p(\boldsymbol{f}) \propto \exp\left\{-\sum_j \gamma_j |f_j|^2\right\}$$

- Full Bayesian: $p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{\theta}_2)\, p(\boldsymbol{\theta})$

$$\boldsymbol{\theta}_1 = \{\sigma_\epsilon^2\}, \quad \boldsymbol{\theta}_2 = \{\gamma_j\}$$

- Inverse Gamma priors for $\boldsymbol{\theta}$
- Joint MAP: $(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}}) = \arg\max_{\boldsymbol{f},\boldsymbol{\theta}} \{p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g})\}$ or
- Marginalizing

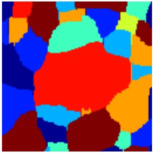
$$p(\boldsymbol{\theta}|\boldsymbol{g}) = \int p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g})\, \mathrm{d}\boldsymbol{f}$$

$$\widehat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \{p(\boldsymbol{\theta}|\boldsymbol{g})\} \longrightarrow \widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{\theta};\boldsymbol{g})\}$$

Gauss-Markov-Potts prior models for images

"In many imaging applications, the objects are, in general, composed of a finite number of materials, and the pixels/voxels corresponding to each materials are grouped in compact regions"

How to model this prior information?



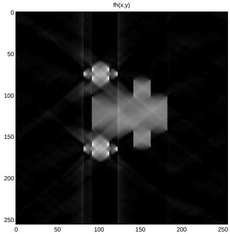
$f(\boldsymbol{r})$ $\qquad z(\boldsymbol{r}) \in \{1, ..., K\}$

$$p(f(\boldsymbol{r})|z(\boldsymbol{r}) = k) = \mathcal{N}(m_k, v_k)$$

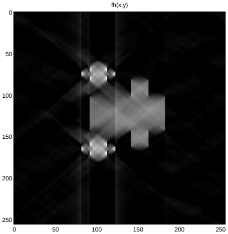
$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{\gamma \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}'))\right\}$$

$$p(\boldsymbol{f}, \boldsymbol{z}, \boldsymbol{\theta}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{\theta}_1)\, p(\boldsymbol{f}|\boldsymbol{z}, \boldsymbol{\theta}_2)\, p(\boldsymbol{z}|\gamma)\, p(\boldsymbol{\theta})$$

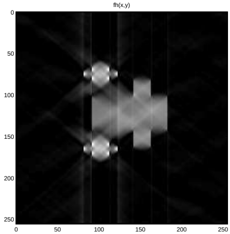
Comparison of different inversion methods on data set 1



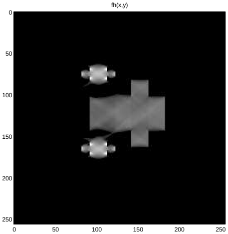
Gerchberg-Papoulis (GP)



Least Squares (LS)



Quad. Reg. (QR)



Proposed method

Multistatic data fusion methods

Method 1: Data Fusion followed by inversion

$$\left.\begin{array}{c} G_1(u,v) \\ M_1(u,v) \\ \\ G_2(u,v) \\ M_2(u,v) \end{array}\right| \longrightarrow \begin{array}{c} G(k_x,k_y) \\ M(k_x,k_y) \end{array} \longrightarrow \boxed{\text{Inversion}} \longrightarrow \widehat{f}(x,y) \longrightarrow \widehat{G}(u,v)$$

with

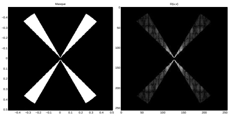
$$G(k_x,k_y) = \begin{cases} (G_1(u,v)+G_2(u,v))/2 & (u,v) \in M_1(u,v) \cap G_2(u,v) \\ G_1(u,v) & (u,v) \in M_1(u,v) \\ G_2(u,v) & (u,v) \in M_2(u,v) \end{cases}$$

and

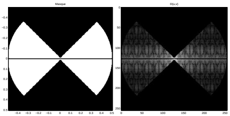
$$M(k_x,k_y) = M_1(u,v) \cup M_2(u,v)$$

Multistatic data fusion methods

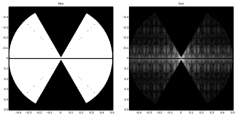
Method 1: Data Fusion followed by inversion



$M_1(u,v)$ $G_1(u,v)$

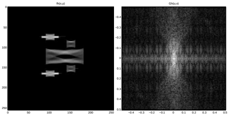


$M_2(u,v)$ $G_2(u,v)$



$M(k_x,k_y)G(k_x,k_y)$

Data Fusion



$\widehat{f}(x,y)$ $\widehat{G}(u,v)$

Inversion

Multistatic data fusion methods

Method 2: Separte inversion followed by image fusion

$$\begin{array}{l} G_1(u,v) \\ M_1(u,v) \end{array} - \boxed{\text{Inversion}} - \widehat{f}_1(x,y)$$

$$|\longrightarrow \boxed{\text{Fusion}} \longrightarrow \widehat{f}(x,y) \longrightarrow \widehat{G}(u,v)$$

$$\begin{array}{l} G_2(u,v) \\ M_2(u,v) \end{array} - \boxed{\text{Inversion}} - \widehat{f}_2(x,y)$$

- Image fusion
  - Coherent addition $\widehat{f}(x,y) = (\widehat{f}_1(x,y) + \widehat{f}_2(x,y))/2$
  - Incoherent addition $\widehat{f}(x,y) = (|\widehat{f}_1(x,y)| + |\widehat{f}_2(x,y)|)/2$
- May need image registration

Multistatic data fusion methods

Method 3: Simultaneous Data Fusion and Inversion

$$\left.\begin{array}{l} G_1(u,v) \\ M_1(u,v) \\ \\ G_2(u,v) \\ M_2(u,v) \end{array}\right\} \longrightarrow \boxed{\begin{array}{c}\text{Fusion}\\ \text{et}\\ \text{Inversion}\end{array}} \longrightarrow \widehat{f}(x,y) \longrightarrow \left\{\begin{array}{l} \widehat{G}_1(u,v) \\ \\ \widehat{G}_2(u,v) \end{array}\right.$$

$$\left\{\begin{array}{l} \boldsymbol{g}_1 = \boldsymbol{H}_1 \boldsymbol{f} + \boldsymbol{\epsilon}_1 \\ \boldsymbol{g}_2 = \boldsymbol{H}_2 \boldsymbol{f} + \boldsymbol{\epsilon}_2 \end{array}\right.$$

Regularization:

$$J(\boldsymbol{f}) = \|\boldsymbol{g}_1 - \boldsymbol{H}_1 \boldsymbol{f}\|^2 + \|\boldsymbol{g}_2 - \boldsymbol{H}_2 \boldsymbol{f}\|^2 + \lambda \|\boldsymbol{D} \boldsymbol{f}\|^2$$

Bayesian Approach for Simultaneous Data Fusion and Inversion

- Joint MAP with sparse Gaussian model:

$$p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g}_1,\boldsymbol{g}_2) \propto p(\boldsymbol{g}_1|\boldsymbol{f},\sigma_{\epsilon_1}^2)\, p(\boldsymbol{g}_2|\boldsymbol{f},\sigma_{\epsilon_2}^2)\, p(\boldsymbol{f}|\{\gamma_j\})\, p(\boldsymbol{\theta})$$

$$\boldsymbol{\theta} = \{\sigma_{\epsilon_1}^2, \sigma_{\epsilon_2}^2, \{\gamma_j\}\}$$

$$(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{f},\boldsymbol{\theta})} \{p(\boldsymbol{f},\boldsymbol{\theta}|\boldsymbol{g}_1,\boldsymbol{g}_2)\}$$

- Gibbs sampling with Gauss-Markov-Potts model:

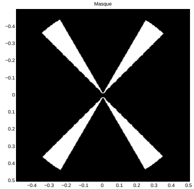
$$p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\boldsymbol{g}_1,\boldsymbol{g}_2) \propto p(\boldsymbol{g}_1|\boldsymbol{f},\sigma_{\epsilon_1}^2)\, p(\boldsymbol{g}_2|\boldsymbol{f},\sigma_{\epsilon_2}^2)\, p(\boldsymbol{f}|\boldsymbol{z},\{m_k,v_k\})\, p(\boldsymbol{\theta})$$

$$\boldsymbol{\theta} = \{\sigma_{\epsilon_1}^2, \sigma_{\epsilon_2}^2, \{m_k,v_k\}\}$$

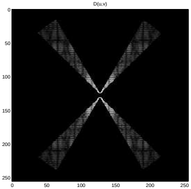
- Note that, in both cases, the estimation of $\boldsymbol{f}$, we optimize:

$$\begin{aligned} J(\boldsymbol{f}) &= -\ln p(\boldsymbol{g}_1|\boldsymbol{f}) - \ln p(\boldsymbol{g}_2|\boldsymbol{f}) - \ln p(\boldsymbol{f}) \\ &= \frac{1}{2\sigma_{\epsilon_1}^2}\|\boldsymbol{g}_1 - \boldsymbol{H}_1\boldsymbol{f}\|^2 + \frac{1}{2\sigma_{\epsilon_2}^2}\|\boldsymbol{g}_2 - \boldsymbol{H}_2\boldsymbol{f}\|^2 + \sum_j \gamma_j f_j^2 \end{aligned}$$

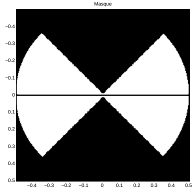
Bayesian Approach for Simultaneous Data Fusion and Inversion



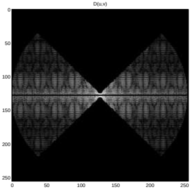
$M_1(u,v)$



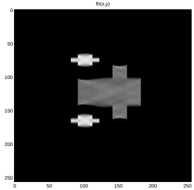
$G_1(u,v)$



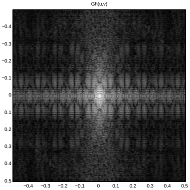
$M_2(u,v)$



$G_2(u,v)$



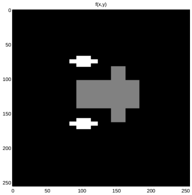
$\widehat{f}(x,y)$



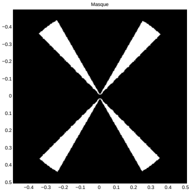
$\widehat{G}(u,v)$

Joint Fusion & Inversion

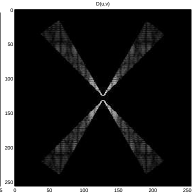
Simulated target and the two data sets



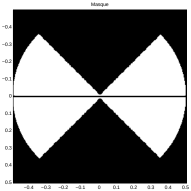
Original target $f(x,y)$



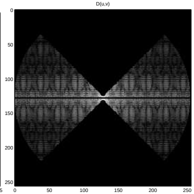
$M_1(u,v)$



$G_1(u,v)$

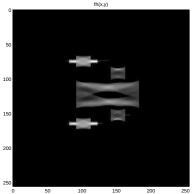


$M_2(u,v)$

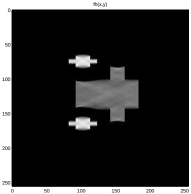


$G_2(u,v)$

Comparison of data fusion methods

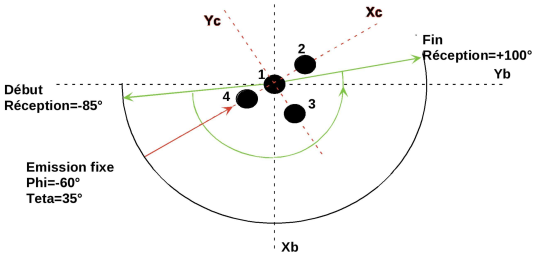


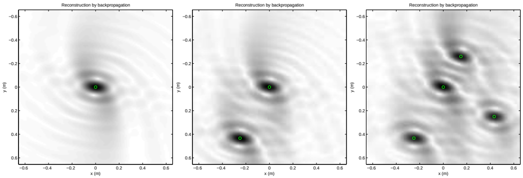
Data Fusion followed by Inversion



Joint Fusion and Inversion

Results on experimental data (Vv polarisation)



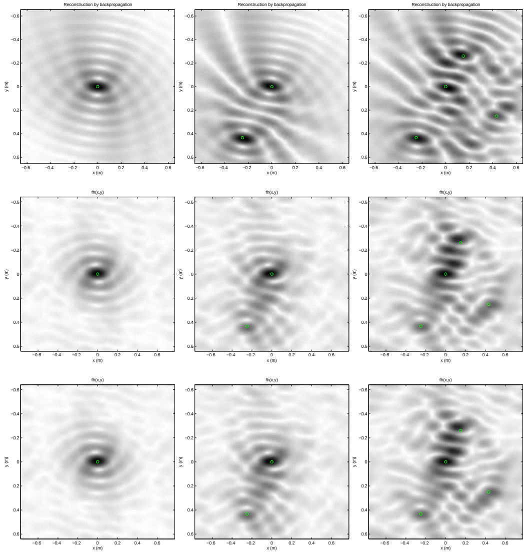


Results on experimental data (2 bands fusion)

BF1 band

BF2 band

BF1 & BF2



Conclusions and Perspectives

- Bayesian estimation framework is an appropriate one for handeling inverse problems and in particular Fusion and inversion of SAR imaging data
- Proposed methods show good results both on simulated and experimental data
- For experimental data, we still need to account for polarisation information
- Present forward modeling assumes a scene with non interacting real point sources
- More accurate forward models are needed for accounting for real scenes:
  Complexe valued, interacting, polarisation, multiple trajectories, ...