Super-Resolution:
A Bayesian approach

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SISO (Single Input Single Output) SR problem

$$?$$
$$\Longrightarrow$$

$g(\boldsymbol{r}) \qquad \longleftarrow \mathcal{B} \longleftarrow \qquad f(\boldsymbol{r})$

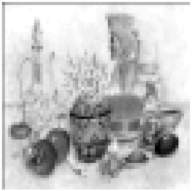
- $\mathcal{B}$: Blurring (needs image restoration)

$$g(x,y) = \iint f(x',y')h(x-x',y-y')\,\mathrm{d}x'\,\mathrm{d}y'$$

$$g(\boldsymbol{r}) = \iint f(\boldsymbol{r}')h(\boldsymbol{r}-\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'$$

- $h(\boldsymbol{r}) = h(x,y)$: Point Spread function
- Convolution/Deconvolution

SISO (Single Input Single Output) SR problem

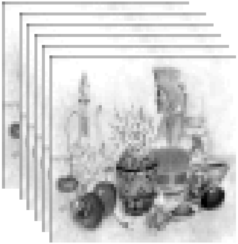


$?$
$\Longrightarrow$

$g(\boldsymbol{r}) \qquad \longleftarrow \mathcal{D}\,\mathcal{B} \longleftarrow \qquad f(\boldsymbol{r})$

- $\mathcal{B}$: Blurring (needs image restoration)
- $\mathcal{D}$: Down sampling (needs interpolation and Up Sampling)

MISO (Multi Input Single Output) SR problem

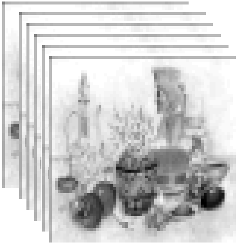


$\overset{?}{\Longrightarrow}$

$$g_k(\boldsymbol{r}) \quad \longleftarrow \mathcal{D}\, \mathcal{M}_k\, \mathcal{B} \longleftarrow \quad f(\boldsymbol{r})$$

- $\mathcal{B}$: Blurring (needs image restoration)
- $\mathcal{M}_k$: Movement
  (needs Registration and image fusion)
- $\mathcal{D}$: Down sampling (needs interpolation and Up Sampling)

MIMO (Multi Input Multi Output) SR problem



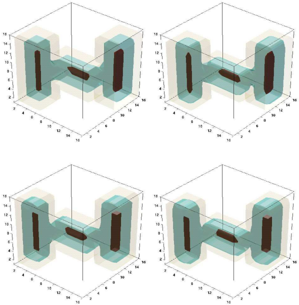
?

$\Longrightarrow$

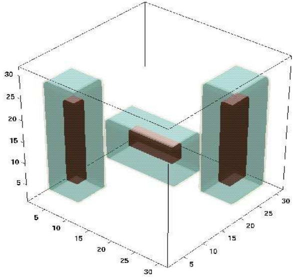
$g_k(\boldsymbol{r}) \quad \longleftarrow \mathcal{D}\; \mathcal{M}_k\; \mathcal{B} \longleftarrow \quad f_k(\boldsymbol{r})$

- $\mathcal{B}$: Blurring (needs image restoration)
- $\mathcal{M}_k$: Movement
  (needs Registration and image fusion)
- $\mathcal{D}$: Down sampling (needs interpolation and Up Sampling)

MISO 3D SR problem



$$\overset{?}{\Longrightarrow}$$



$$g_k(\boldsymbol{r}) \quad \longleftarrow \mathcal{D}\mathcal{M}_k\mathcal{B} \longleftarrow \quad f(\boldsymbol{r})$$

- Non Destructive Testing (NDT) using Computed Tomography (CT)
- Multi modal medical imaging

Forward model for SISO SR problem

$\overset{?}{\Longrightarrow}$

$g(\boldsymbol{r})$ $\longleftarrow \mathcal{B} \longleftarrow$ $f(\boldsymbol{r})$

- $\mathcal{B}$: Blurring (needs image restoration)

$$g(x,y) = \iint f(x',y')h(x-x',y-y')\,\mathrm{d}x'\,\mathrm{d}y'$$

$$g(\boldsymbol{r}) = \iint f(\boldsymbol{r}')h(\boldsymbol{r}-\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'$$

- $h(\boldsymbol{r}) = h(x,y)$: Point Spread function
- Convolution/Deconvolution

Image Restoration (Deconvolution)

- Forward problem: Convolution:

$$g(x,y) = \iint f(x',y')h(x-x',y-y')\,\mathrm{d}x'\,\mathrm{d}y'$$

- Inverse problem: Given g and h find f: Deconvolution
- Fourier based methods:

$$\begin{cases} F(u,v) = \iint f(x,y)\exp\left\{-j(ux+vy)\right\}\,\mathrm{d}x\,\mathrm{d}y \\ f(x,y) = \iint F(u,v)\exp\left\{+j(ux+vy)\right\}\,\mathrm{d}u\,\mathrm{d}v \end{cases}$$

$$g(x,y) = f(x,y) * h(x,y) \longrightarrow G(u,v) = H(u,v)F(u,v)$$

- Inverse Filtering: $F(u,v) = \frac{1}{H(u,v)}G(u,v)$

Deconvolution: 1D and 2D cases

$$\begin{array}{ccc} & & \epsilon(t) \\ & & \downarrow \\ f(t) \longrightarrow & \boxed{\;h(t)\;} & \longrightarrow \bigoplus \longrightarrow g(t) \end{array}$$

$$g(t) = \int f(t')\, h(t - t')\, \mathrm{d}t' + \epsilon(t)$$

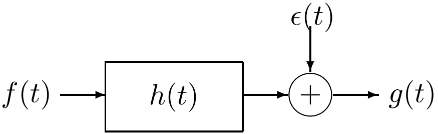
- $f(t)$, $g(t)$ and $\epsilon(t)$ are modelled as Gaussian random signal

$$\begin{array}{ccc} & & \epsilon(x,y) \\ & & \downarrow \\ f(x,y) \longrightarrow & \boxed{\;h(x,y)\;} & \longrightarrow \bigoplus \longrightarrow g(x,y) \end{array}$$

$$g(x,y) = \iint f(x',y')\, h(x - x', y - y')\, \mathrm{d}x'\, \mathrm{d}y' + \epsilon(x,y)$$

- $f(x,y)$, $g(x,y)$ and $\epsilon(x,y)$ are modelled as homogeneous and Gaussian random fields

Wiener Filtering: 1D Case



$$g(t) = h(t) * f(t) + \epsilon(t)$$

- Expected values:

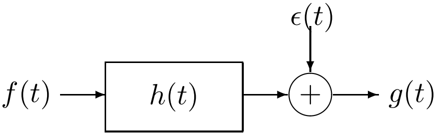
$$\mathsf{E}\{g(t)\} = h(t) * \mathsf{E}\{f(t)\} + \mathsf{E}\{\epsilon(t)\}$$

- Auto and Inter correlation functions:

$$R_{gg}(\tau) = \mathsf{E}\{g(t)\, g(t+\tau)\}$$
$$R_{ff}(\tau) = \mathsf{E}\{f(t)\, f(t+\tau)\}$$

$$R_{\epsilon f}(\tau) = R_{f\epsilon}(-\tau) = \mathsf{E}\{\epsilon(t)\, f(t+\tau)\}$$
$$R_{gf}(\tau) = R_{fg}(-\tau) = \mathsf{E}\{g(t)\, f(t+\tau)\}$$

Wiener Filtering: 1D Case



- Auto and Inter correlation functions:

$$
\begin{aligned}
R_{gg}(\tau) &= h(t) * h(t) * R_{ff}(\tau) + R_{\epsilon\epsilon}(\tau) \\
R_{gf}(\tau) &= h(t) * R_{ff}(\tau)
\end{aligned}
$$

- Spectral density functions:

$$
\begin{aligned}
S_{gg}(\omega) &= |H(\omega)|^2 S_{ff}(\omega) + R_{\epsilon\epsilon}(\omega) \\
S_{gf}(\omega) &= H(\omega) S_{ff}(\omega) \\
S_{fg}(\omega) &= H^*(\omega) S_{ff}(\omega)
\end{aligned}
$$

- Wiener filtering:

$g(t) \longrightarrow \boxed{w(t)} \longrightarrow \widehat{f}(t)$ or $G(\omega) \longrightarrow \boxed{W(\omega)} \longrightarrow \widehat{F}(\omega)$

Wiener Filtering: 1D Case

$$EQM = \mathsf{E}\left\{[f(t) - \widehat{f}(t)]^2\right\} = \mathsf{E}\left\{[f(t) - w(t) * g(t)]^2\right\}$$

$$\frac{\partial EQM}{\partial f} = -2\mathsf{E}\left\{[f(t) - w(t) * g(t)] * g(t+\tau)\right\} = 0$$

$$\mathsf{E}\left\{[f(t) - w(t) * g(t)]\, g(t+\tau)\right\} = 0 \quad \forall t, \tau \longrightarrow$$

$$R_{fg}(\tau) = w(t) * R_{gg}(\tau)$$

$$W(\omega) = \frac{S_{fg}(\omega)}{S_{gg}(\omega)} = \frac{H^*(\omega)\, S_{ff}(\omega)}{|H(\omega)|^2\, S_{ff}(\omega) + S_{\epsilon\epsilon}(\omega)}$$

$$\boxed{W(\omega) = \frac{H^*(\omega) S_{ff}(\omega)}{|H(\omega)|^2 S_{ff}(\omega) + S_{\epsilon\epsilon}(\omega)} = \frac{1}{H(\omega)} \frac{|H(\omega)|^2}{|H(\omega)|^2 + \frac{S_{\epsilon\epsilon}(\omega)}{S_{ff}(\omega)}}}$$

Wiener Filtering: 2D Case

- Linear Estimation: $\widehat{f}(x,y)$ is such that:
  - $\widehat{f}(x,y)$ depends on $g(x,y)$ in a linear way:

$$\widehat{f}(x,y) = \iint g(x',y')\, w(x-x',y-y')\, \mathrm{d}x'\, \mathrm{d}y'$$

    $w(x,y)$ is the impulse response of the Wiener filtre

  - minimizes MSE: $\mathrm{E}\left\{|f(x,y)-\widehat{f}(x,y)|^2\right\}$
- Orthogonality condition:

$$(f(x,y)-\widehat{f}(x,y)) \perp g(x',y') \quad \longrightarrow \quad \mathrm{E}\left\{(f(x,y)-\widehat{f}(x,y))\, g(x',y')\right\} = 0$$

$$\widehat{f} = g * w \;\longrightarrow\; \mathrm{E}\left\{(f(x,y)-g(x,y)*w(x,y))\, g(x+\alpha_1,y+\alpha_2)\right\} = 0$$

$$R_{fg}(\alpha_1,\alpha_2) = (R_{gg}*w)(\alpha_1,\alpha_2) \longrightarrow \mathrm{TF} \longrightarrow S_{fg}(u,v) = S_{gg}(u,v)W(u,v)$$

$$\Downarrow$$

$$\boxed{W(u,v) = \frac{S_{fg}(u,v)}{S_{gg}(u,v)}}$$

Wiener filtering: 1D and 2D Cases

Signal	Image
$W(\omega)=\frac{S_{fg}(\omega)}{S_{gg}(\omega)}$	$W(u,v)=\frac{S_{fg}(u,v)}{S_{gg}(u,v)}$

Particular Case:

$f(x,y)$ and $\epsilon(x,y)$ are assumed to be centered and non correlated

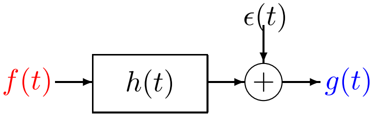
$$S_{fg}(u,v)=H'(u,v)\,S_{ff}(u,v)$$

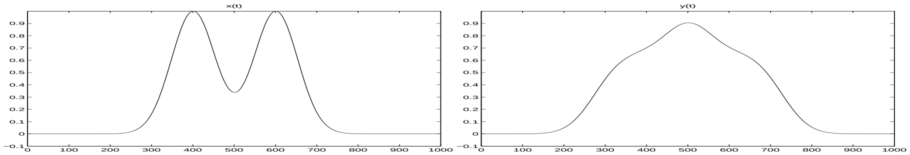
$$S_{gg}(u,v)=|H(u,v)|^2\,S_{ff}(u,v)+S_{\epsilon\epsilon}(u,v)$$

$$W(u,v)=\frac{H'(u,v)S_{ff}(u,v)}{|H(u,v)|^2S_{ff}(u,v)+S_{\epsilon\epsilon}(u,v)}$$

Signal	Image
$W(\omega)=\frac{1}{H(\omega)}\frac{\|H(\omega)\|^2}{\|H(\omega)\|^2+\frac{S_{\epsilon\epsilon}(\omega)}{S_{ff}(\omega)}}$	$W(u,v)=\frac{1}{H(u,v)}\frac{\|H(u,v)\|^2}{\|H(u,v)\|^2+\frac{S_{\epsilon\epsilon}(u,v)}{S_{ff}(u,v)}}$

Convolution, Deconvolution, Identification and Blind Deconvolution in signal processing

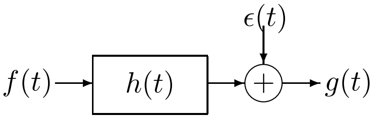




$$g(t) = \int f(t')h(t-t')\,\mathrm{d}t' + b(t) = \int h(t')f(t-t')\,\mathrm{d}t' + b(t)$$

- Convolution: Given f and h compute g
- Identification: Given f and g estimate h
- Deconvolution: Given g and h estimate f
- Blind deconvolution: Given g estimate both h and f

Convolution: Discretization



$$g(t) = \int f(t')\, h(t-t')\, \mathrm{d}t' + \epsilon(t) = \int h(t')\, f(t-t')\, \mathrm{d}t' + \epsilon(t)$$

- The signals $f(t)$, $g(t)$, $h(t)$ are discretized with the same sampling period $\Delta T = 1$,
- The impulse response is finite (FIR) : $h(t) = 0$, for t such that $t < -q\Delta T$ or $\forall t > p\Delta T$.

$$g(m) = \sum_{k=-q}^{p} h(k)\, f(m-k) + \epsilon(m), \quad m = 0, \cdots, M$$

Convolution: Discretized matrix vector forms

$$
\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \end{bmatrix}
=
\begin{bmatrix}
h(p) & \cdots & h(0) & \cdots & h(-q) & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & & \ddots & & \ddots & & & & \vdots \\
\vdots & & & & & & & & & \vdots \\
\vdots & & h(p) & \cdots & h(0) & \cdots & h(-q) & & & \vdots \\
\vdots & & & & & & & & & \vdots \\
\vdots & & & \ddots & & & \ddots & & & \vdots \\
\vdots & & & & \ddots & & & & & 0 \\
0 & \cdots & \cdots & & 0 & h(p) & \cdots & h(0) & \cdots & h(-q)
\end{bmatrix}
\begin{bmatrix} f(-p) \\ \vdots \\ f(0) \\ f(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f(M) \\ f(M+1) \\ \vdots \\ f(M+q) \end{bmatrix}
$$

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- $\boldsymbol{g}$ is a $(M+1)$-dimensional vector,
- $\boldsymbol{f}$ has dimension $M+p+q+1$,
- $\boldsymbol{h} = [h(p), \cdots, h(0), \cdots, h(-q)]$ has dimension $(p+q+1)$
- $\boldsymbol{H}$ has dimensions $(M+1) \times (M+p+q+1)$.

Convolution: Discretized matrix vector form

- If system is causal ($q=0$) we obtain

$$\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \end{bmatrix} = \begin{bmatrix} h(p) & \cdots & h(0) & 0 & \cdots & \cdots & 0 \\ 0 & & & & & & \vdots \\ \vdots & & & & & & \vdots \\ \vdots & & h(p) & \cdots & h(0) & & \vdots \\ \vdots & & & & & & \vdots \\ \vdots & & & & & & 0 \\ 0 & \cdots & \cdots & 0 & h(p) & \cdots & h(0) \end{bmatrix} \begin{bmatrix} f(-p) \\ \vdots \\ f(0) \\ f(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f(M) \end{bmatrix}$$

- $\boldsymbol{g}$ is a $(M+1)$-dimensional vector,
- $\boldsymbol{f}$ has dimension $M+p+1$,
- $\boldsymbol{h} = [h(p), \cdots, h(0)]$ has dimension $(p+1)$
- $\boldsymbol{H}$ has dimensions $(M+1) \times (M+p+1)$.

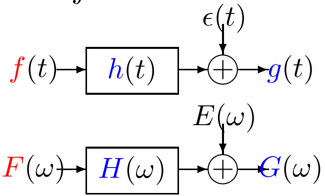
Convolution: Causal systems and causal input

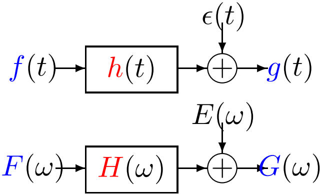
$$\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \end{bmatrix} = \begin{bmatrix} h(0) & & & & & \\ h(1) & \ddots & & & & \\ \vdots & & & & & \\ h(p) & \cdots & & h(0) & & \\ 0 & \ddots & & & \ddots & \\ \vdots & & & & & \\ 0 & \cdots & 0 & h(p) & \cdots & h(0) \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f(M) \end{bmatrix}$$

- $\boldsymbol{g}$ is a $(M+1)$-dimensional vector,
- $\boldsymbol{f}$ has dimension $M+1$,
- $\boldsymbol{h} = [h(p), \cdots, h(0)]$ has dimension $(p+1)$
- $\boldsymbol{H}$ has dimensions $(M+1) \times (M+1)$.

Convolution, Identification, Deconvolution and Blind deconvolution problems

$$g(t)=\int f(t')\,h(t-t')\,\mathrm{d}t'+\epsilon(t)=\int h(t')\,f(t-t')\,\mathrm{d}t'+\epsilon(t)$$





$$G(\omega)=H(\omega)\,F(\omega)+E(\omega) \qquad G(\omega)=H(\omega)\,F(\omega)+E(\omega)$$

$$F(\omega)=\frac{G(\omega)}{H(\omega)}+\frac{E(\omega)}{H(\omega)} \qquad H(\omega)=\frac{G(\omega)}{F(\omega)}+\frac{E(\omega)}{F(\omega)}$$

- Convolution: Given h and f compute g
- Identification: Given f and g estimate h
- Simple Deconvolution: Given h and g estimate f
- Blind Deconvolution: Given g estimate h and f

Deconvolution: Given g and h estimate f

- Direct computation: `f=deconv(g,h)`
- Fourier domain: Inverse Filtering $F(\omega) = \frac{G(\omega)}{H(\omega)}$
  - Compute $H(\omega)$, $G(\omega)$ and $F(\omega) = \frac{G(\omega)}{H(\omega)}$
  - Compute $g(t)$ by inverse FT of $F(\omega)$
- Main difficulties: Divide by zero and noise amplification

Identification: Given g and f estimate h

- Direct computation:
  - $f(t) = \delta(t) \longrightarrow g(t) = h(t) \longrightarrow h(t) = g(t)$
  - $f(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \longrightarrow g(t) = \int_0^t h(t) \, \mathrm{d}t \longrightarrow h(t) = \frac{\mathrm{d}g(t)}{\mathrm{d}t}$
- Fourier domain: Inverse Filtering $H(\omega) = \frac{G(\omega)}{F(\omega)}$
  - Compute $F(\omega)$, $G(\omega)$ and $H(\omega) = \frac{G(\omega)}{F(\omega)}$
  - Compute $h(t)$ by inverse FT of $H(\omega)$
- Main difficulties: Divide by zero and noise amplification

Convolution: Discretization for Identification

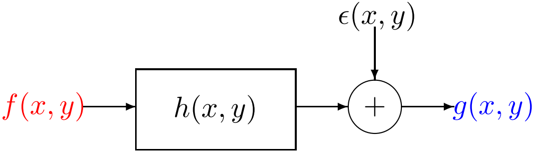
Causal systems and causal input

$$
\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g(M) \end{bmatrix}
=
\begin{bmatrix}
0 & & . & 0 & f(0) \\
. & & . & f(0) & f(1) \\
. & & f(0) & f(1) & \vdots \\
. & . & . & . & \vdots \\
f(0) & f(1) & & & f(M-p) \\
f(1) & . & & . & . \\
. & & & . & . \\
. & & . & & . \\
. & . & & & . \\
f(M-p) & . & . & . & f(M)
\end{bmatrix}
\begin{bmatrix} h(p) \\ h(p-1) \\ \vdots \\ \vdots \\ h(1) \\ h(0) \end{bmatrix}
$$

$$\boldsymbol{g} = \boldsymbol{F}\,\boldsymbol{h} + \boldsymbol{\epsilon}$$

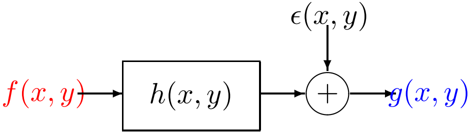
- $\boldsymbol{g}$ is a $(M+1)$-dimensional vector,
- $\boldsymbol{F}$ has dimension $(M+1) \times (p+1)$,

Convolution in imaging systems



$$g(x, y) = \iint f(x', y')\, h(x, y; x', y')\, \mathrm{d}x'\, \mathrm{d}y' + \epsilon(x, y)$$

2D Convolution for image restoration



$$g(x,y) = \iint_D f(x',y')\, h(x-x', y-y')\; \mathrm{d}x'\; \mathrm{d}y' + b(x,y)$$

$$g(m\Delta x, n\Delta y) = \sum_{i=-I}^{+I} \sum_{j=-J}^{+J} h(i\Delta x, j\Delta y) f((m-i)\Delta x, (n-j)\Delta y)$$

$$\left\{ \begin{bmatrix} m = 1,\ldots,M \\ n = 1,\ldots,N \end{bmatrix} \right. \qquad \Delta x = \Delta y = 1$$

$$g(m,n) = \sum_{i=-I}^{+I} \sum_{j=-J}^{+J} h(i,j)\, f(m-i, n-j)$$

2D Convolution for image restoration

Two caracteristics:

$$g(m,n) = \sum_{i=-I}^{+I} \sum_{j=-J}^{+J} h(i,j)\, f(m-i, n-j)$$

- $g(m,n)$ depends on $f(k,l)$ for $(k,l) \in \mathcal{N}(k,l)$ where $\mathcal{N}(k,l)$ means the neigborhood pixels around the pixel $'(k,l) \longrightarrow$ No Causality
- The boarding effects cannot be neglected as easily as in the 1D case.

Vectorial Forme:

$$g(m,n) = \sum_{i=-I}^{+I} \sum_{j=-J}^{+J} h(i,j)\, f(m-i, n-j)$$

$$g(m,n) \left\{ \begin{bmatrix} m = 1, \ldots, M \\ n = 1, \ldots, N \end{bmatrix} \right. \quad f(k,l) \left\{ \begin{bmatrix} k = 1, \ldots, K \\ l = 1, \ldots, L \end{bmatrix} \right. \quad h(i,j) \left\{ \begin{bmatrix} i = 1, \ldots, I \\ j = 1, \ldots, J \end{bmatrix} \right.$$

$$\boldsymbol{g} = \boldsymbol{H} \boldsymbol{f}$$

2D Convolution for image restoration

$$\boldsymbol{g} = [\quad \underset{---1---}{g_{(1,1)}, \ldots, g(M,1)}, \quad \underset{---2---}{g_{(1,2)}, \ldots, g_{(M,2)}}, \quad \ldots, \quad \underset{---N---}{g_{(1,N)}, \ldots, g_{(M,N)}}]^t$$

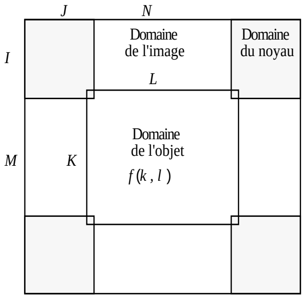
$$\boldsymbol{f} = [\quad \underset{---1---}{f_{(1,1)}, \ldots, f_{(K,1)}}, \quad \underset{---2---}{f_{(1,2)}, \ldots, f(K,2)}, \quad \ldots, \quad \underset{---L---}{f_{(1,L)}, \ldots, f_{(K,L)}}]^t$$

The structure of the matrix $\boldsymbol{H}$ depends on the domaines D_h, D_f and D_g .

Matrix Form $\boldsymbol{H}$:

- Image $>$ Object
- Image$=$Object
- Image $<$ Object

2D Convolution for image restoration: Image > Object



$$D_g > D_f \quad \left\{ \begin{bmatrix} M = K + I - 1 \\ N = L + J - 1 \end{bmatrix} \right.$$

$$g(m,n) \left\{ \begin{bmatrix} m = 1,\ldots,M \\ n = 1,\ldots,N \end{bmatrix} \right. \quad f(k,l) \left\{ \begin{bmatrix} k = 1,\ldots,K \\ l = 1,\ldots,L \end{bmatrix} \right. \quad h(i,j) \left\{ \begin{bmatrix} i = 1,\ldots,I \\ j = 1,\ldots,J \end{bmatrix} \right.$$

$$\boldsymbol{g} = [\quad \underbrace{g_{(1,1)},\ldots,g(M,1)}_{---1---}, \quad \underbrace{g_{(1,2)},\ldots,g_{(M,2)}}_{---2---}, \quad \ldots, \quad \underbrace{g_{(1,N)},\ldots,g_{(M,N)}}_{---N---}]^t$$

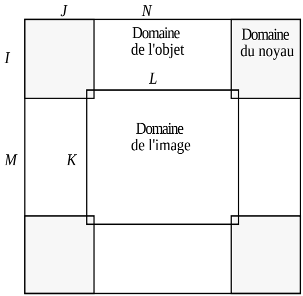
$$\boldsymbol{f} = [\quad \underbrace{f_{(1,1)},\ldots,f_{(K,1)}}_{---1---}, \quad \underbrace{f_{(1,2)},\ldots,f(K,2)}_{---2---}, \quad \ldots, \quad \underbrace{f_{(1,L)},\ldots,f_{(K,L)}}_{---L---}]^t$$

2D Convolution for image restoration: Image > Object

$$
\boldsymbol{H} = \begin{bmatrix}
H_I & \cdot & \cdots & \cdots & \cdots & \cdot \\
\vdots & H_I & \ddots & & & \vdots \\
\vdots & & & \ddots & & \vdots \\
\vdots & & & & \ddots & \vdots \\
H_1 & & & & H_I & \cdot \\
\cdot & H_1 & & & & H_I \\
\vdots & \ddots & H_1 & & & \vdots \\
\vdots & & \ddots & & & \vdots \\
\vdots & & & \ddots & & \vdots \\
\cdot & \cdots & \cdots & \cdots & \cdot & H_1
\end{bmatrix}
\quad
H_i = \begin{bmatrix}
h(i,J) & \cdot & \cdots & \cdots & \cdots & \cdot \\
\vdots & h(i,J) & \ddots & & & \vdots \\
\vdots & & & \ddots & & \vdots \\
\vdots & & & & \ddots & \vdots \\
h(i,1) & & & & h(i,J) & \cdot \\
\cdot & h(i,1) & & & & h(i,J) \\
\vdots & \ddots & h(i,1) & & & \vdots \\
\vdots & & \ddots & & & \vdots \\
\vdots & & & \ddots & & \vdots \\
\cdot & \cdots & \cdots & \cdots & \cdot & h(i,1)
\end{bmatrix}
$$

Toeplitz-Bloc-Toeplitz

2D Convolution for image restoration: Image $<$ Object



$$D_g < D_f \quad \left\{ \begin{bmatrix} M = K - I - 1 \\ N = L - J + 1 \end{bmatrix} \right.$$

$$g(m,n) \left\{ \begin{bmatrix} m = 1, \ldots, M \\ n = 1, \ldots, N \end{bmatrix} \right. \qquad f(k,l) \left\{ \begin{bmatrix} k = 1, \ldots, K \\ l = 1, \ldots, L \end{bmatrix} \right. \qquad h(i,j) \left\{ \begin{bmatrix} i = 1, \ldots, I \\ j = 1, \ldots, J \end{bmatrix} \right.$$

$$\boldsymbol{g} = [\quad \underset{---1---}{g_{(1,1)}, \ldots, g_{(M,1)}}, \quad \underset{---2---}{g_{(1,2)}, \ldots, g_{(M,2)}}, \quad \ldots, \quad \underset{---N---}{g_{(1,N)}, \ldots, g_{(M,N)}}]^t$$

$$\boldsymbol{f} = [\quad \underset{---1---}{f_{(1,1)}, \ldots, f_{(K,1)}}, \quad \underset{---2---}{f_{(1,2)}, \ldots, f_{(K,2)}}, \quad \ldots, \quad \underset{---L---}{f_{(1,L)}, \ldots, f_{(K,L)}}]^t$$

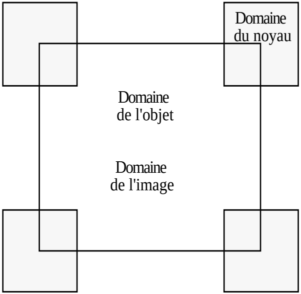
2D Convolution for image restoration: Image $<$ Object

$$
\boldsymbol{H} = \begin{bmatrix}
H_1 & H_2 & \cdots & H_I & \cdot & \cdots & \cdot \\
\cdot & H_1 & & & H_I & \ddots & \vdots \\
\vdots & \ddots & H_1 & & & H_I & \cdot \\
\vdots & & & & & & \vdots \\
\cdot & \cdots & \cdot & H_1 & H_2 & & H_I
\end{bmatrix}
$$

with

$$
H_i = \begin{bmatrix}
h(i,1) & h(i,2) & \cdots & h(i,J) & \cdot & \cdots & \cdot \\
\cdot & h(i,1) & & & h(i,J) & \ddots & \vdots \\
\vdots & \ddots & h(i,1) & & & h(i,J) & \cdot \\
\vdots & & & & & & \vdots \\
\cdot & \cdots & \cdot & h(i,1) & h(i,2) & & h(i,J)
\end{bmatrix}
$$

2D Convolution for image restoration: Image=Object



$$D_g = D_f \quad \left\{ \begin{bmatrix} M = K \\ N = L \end{bmatrix} \right.$$

$$g(m,n) \left\{ \begin{bmatrix} m = 1,\ldots,M \\ n = 1,\ldots,N \end{bmatrix} \right. \quad f(k,l) \left\{ \begin{bmatrix} k = 1,\ldots,K \\ l = 1,\ldots,L \end{bmatrix} \right. \quad h(i,j) \left\{ \begin{bmatrix} i = 1,\ldots,I \\ j = 1,\ldots,J \end{bmatrix} \right.$$

$$\boldsymbol{g} = [\quad \underset{---1---}{g_{(1,1)},\ldots,g_{(M,1)}}, \quad \underset{---2---}{g_{(1,2)},\ldots,g_{(M,2)}}, \quad \ldots, \quad \underset{---N---}{g_{(1,N)},\ldots,g_{(M,N)}}]^t$$

$$\boldsymbol{f} = [\quad \underset{---1---}{f_{(1,1)},\ldots,f_{(K,1)}}, \quad \underset{---2---}{f_{(1,2)},\ldots,f_{(K,2)}}, \quad \ldots, \quad \underset{---L---}{f_{(1,L)},\ldots,f_{(K,L)}}]^t$$

2D Convolution for image restoration: Circulante forme

$$g(m,n)\left\{\begin{bmatrix} m=1,\ldots,M \\ n=1,\ldots,N \end{bmatrix}\right. \quad f(k,l)\left\{\begin{bmatrix} k=1,\ldots,K \\ l=1,\ldots,L \end{bmatrix}\right. \quad h(i,j)\left\{\begin{bmatrix} i=1,\ldots,I \\ j=1,\ldots,J \end{bmatrix}\right.$$

$$\left\{\begin{bmatrix} P=K+I-1 \\ Q=L+J-1 \end{bmatrix}\right.$$

$$\tilde{f}(m,n)=\begin{bmatrix} f((m,n)) & \vdots & 0 \\ \cdots\cdots & \cdots & \cdots\cdots \\ 0 & \vdots & 0 \end{bmatrix} \qquad \dim(\tilde{f})=[P,Q]$$

$$\tilde{g}(k,l)=\begin{bmatrix} g(k,l) & \vdots & 0 \\ \cdots\cdots & \cdots & \cdots\cdots \\ 0 & \vdots & 0 \end{bmatrix} \qquad \dim(\tilde{g})=[P,Q]$$

$$\tilde{h}(i,j)=\begin{bmatrix} h(i,j) & \vdots & 0 \\ \cdots\cdots & \cdots & \cdots\cdots \\ 0 & \vdots & 0 \end{bmatrix} \qquad \dim(\tilde{h})=[P,Q]$$

2D Convolution for image restoration: Circulante forme

$$H = \begin{bmatrix} H_1 & H_2 & \cdots & \cdots & \cdots & H_P \\ H_P & H_1 & H_2 & \cdots & \cdots & H_{P-1} \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ H_P & H_{P-1} & \cdots & \cdots & \cdots & H_1 \end{bmatrix} \quad \text{bloc-circulante}$$

$$H_i = \begin{bmatrix} h(i,1) & h(i,2) & \cdots & \cdots & \cdots & h(i,P) \\ h(i,P) & h(i,1) & h(i,2) & \cdots & \cdots & h(i,P) \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ h(i,P) & h(i,P-1) & h(i,P-2) & \cdots & \cdots & h(i,1) \end{bmatrix} \quad \text{circulante}$$

Circulante-Bloc-Circulante

Classification of the signal and image restoration methods

- **Analytical methods**

$$f(t) \longrightarrow \boxed{\quad \mathcal{H} \quad} \longrightarrow g(t) = \mathcal{H}[f(t)]$$

$$f(x,y) \longrightarrow \boxed{\quad \mathcal{H} \quad} \longrightarrow g(x,y) = \mathcal{H}[f(x,y)]$$

$\mathcal{H}$ Linear Operator

$$g(t) \longrightarrow \boxed{\quad \mathcal{G} \quad} \longrightarrow \widehat{f}(t) = \mathcal{H}^{-1}[f(t)]$$

$$g(x,y) \longrightarrow \boxed{\quad \mathcal{G} \quad} \longrightarrow \widehat{f}(x,y) = \mathcal{H}^{-1}[f(x,y)]$$

$\mathcal{G}$ Linear Operator approximating $\mathcal{H}^{-1}$

  - Inverse Filtring
  - Pseudo–inverse Filtering
  - Wiener Filtering

Classification of the signal and image restoration methods

- **Algebraic methods**

$$g(t) = \mathcal{H}[f(t)] \longrightarrow \text{ Discretization } \longrightarrow \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f}$$

$$g(x,y) = \mathcal{H}[f(x,y)] \longrightarrow \text{ Discretization } \longrightarrow \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f}$$

Ideal case : $\boldsymbol{H}$ invertible $\longrightarrow \widehat{\boldsymbol{f}} = \boldsymbol{H}^{-1}\boldsymbol{g}$
More general case : $\boldsymbol{H}$ is not invertible

  - Generalized Inversion
  - Least Squares (LS) and Minimum norm LS
  - Regularization

- **Probabilistic methods**
  - Wiener Filtering
  - Kalman Filtering
  - General Bayesian approach

Algebraic Approches

Signal	Image
$f(t) \longrightarrow \boxed{h(t)} \longrightarrow g(t)$	$f(x,y) \longrightarrow \boxed{h(x,y)} \longrightarrow g(x,y)$

Discretization

$\Downarrow$

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f}$$

- **Ideal case:** $\boldsymbol{H}$ invertible $\longrightarrow \widehat{\boldsymbol{f}} = \boldsymbol{H}^{-1}\boldsymbol{g}$
- $M > N$ **Least Squares**: $\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\}$

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 = [\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}]'[\boldsymbol{g} - \boldsymbol{H}\widehat{\boldsymbol{f}}]$$

$$\nabla J = -2\boldsymbol{H}'[\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}] = 0 \longrightarrow \boldsymbol{H}'\boldsymbol{H}\boldsymbol{f} = \boldsymbol{H}'\boldsymbol{g} \longrightarrow \widehat{\boldsymbol{f}} = (\boldsymbol{H}'\boldsymbol{H})^{-1}\boldsymbol{H}'\boldsymbol{g}$$

- $M < N$ **Min Norme Solution**: $\widehat{\boldsymbol{f}} = \boldsymbol{H}'(\boldsymbol{H}\boldsymbol{H}')^{-1}\boldsymbol{g}$

Regularization

- $M > N$ **Least Squares**: $\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2\}$
- $M < N$ **Min Norme Solution**: $\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{H}\boldsymbol{f}=\boldsymbol{g}} \{\|\boldsymbol{f}\|^2\}$
- **Regularization**: $\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{H}\boldsymbol{f}=\boldsymbol{g}} \{\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{f}\|^2\}$

$$J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}]\|^2 + \lambda\|\boldsymbol{D}\boldsymbol{f}\|^2$$

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 & \cdots & & \cdots & 0 \\ -1 & 1 & \ddots & & & \vdots \\ 0 & -1 & 1 & \ddots & & \vdots \\ & 0 & -1 & 1 & \ddots & \\ & & & & & \\ 0 & & & 0 & -1 & 1 \end{bmatrix} \text{ or } \boldsymbol{D} = \begin{bmatrix} 1 & 0 & \cdots & & \cdots & 0 \\ -2 & 1 & \ddots & & & \vdots \\ 1 & -2 & 1 & \ddots & & \vdots \\ & 1 & -2 & 1 & \ddots & \\ & & & & & \\ 0 & & & 1 & -2 & 1 \end{bmatrix}$$

$$\nabla J = 2\boldsymbol{H}'[\boldsymbol{H}\boldsymbol{f} - \boldsymbol{g}]' + 2\lambda\boldsymbol{D}'\boldsymbol{D}\boldsymbol{f} = 0$$

$$[\boldsymbol{H}'\boldsymbol{H} + \lambda\boldsymbol{D}'\boldsymbol{D}]\widehat{\boldsymbol{f}} = \boldsymbol{H}'\boldsymbol{g} \longrightarrow \widehat{\boldsymbol{f}} = [\boldsymbol{H}'\boldsymbol{H} + \lambda\boldsymbol{D}'\boldsymbol{D}]^{-1}\boldsymbol{H}'\boldsymbol{g}$$

Bayesian estimation approach

$$\mathcal{M}: \qquad \boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Observation model $\mathcal{M}$ + Hypothesis on the noise $\boldsymbol{\epsilon} \longrightarrow$
  $p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M}) = p_{\boldsymbol{\epsilon}}(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f})$
- A priori information $\quad p(\boldsymbol{f}|\mathcal{M})$
- Bayes : $\quad p(\boldsymbol{f}|\boldsymbol{g};\mathcal{M}) = \dfrac{p(\boldsymbol{g}|\boldsymbol{f};\mathcal{M})\,p(\boldsymbol{f}|\mathcal{M})}{p(\boldsymbol{g}|\mathcal{M})}$

Link with regularization :

Maximum A Posteriori (MAP) :

$$\begin{aligned}
\widehat{\boldsymbol{f}} &= \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{g}|\boldsymbol{f})\; p(\boldsymbol{f})\} \\
&= \arg\min_{\boldsymbol{f}} \{-\ln p(\boldsymbol{g}|\boldsymbol{f}) - \ln p(\boldsymbol{f})\}
\end{aligned}$$

with $\quad Q(\boldsymbol{g},\boldsymbol{H}\boldsymbol{f}) = -\ln p(\boldsymbol{g}|\boldsymbol{f}) \quad$ and $\quad \lambda\Omega(\boldsymbol{f}) = -\ln p(\boldsymbol{f})$

Case of linear models and Gaussian priors

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{\epsilon}$$

- Hypothesis on the noise: $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma_\epsilon^2 \boldsymbol{I})$

$$p(\boldsymbol{g}|\boldsymbol{f}) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2\right\}$$

- Hypothesis on $\boldsymbol{f}$: $\boldsymbol{f} \sim \mathcal{N}(0, \sigma_f^2(\boldsymbol{D}'\boldsymbol{D})^{-1})$

$$p(\boldsymbol{f}) \propto \exp\left\{-\frac{1}{2\sigma_f^2}\|\boldsymbol{D}\boldsymbol{f}\|^2\right\}$$

- A posteriori:

$$\begin{aligned} p(\boldsymbol{f}|\boldsymbol{g}) & \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 - \frac{1}{2\sigma_f^2}\|\boldsymbol{D}\boldsymbol{f}\|^2\right\} \\ & \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}(\boldsymbol{f} - \widehat{\boldsymbol{f}})'\widehat{\boldsymbol{\Sigma}}^{-1}(\boldsymbol{f} - \widehat{\boldsymbol{f}})\right\} \end{aligned}$$

- MAP : $\quad \widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \{p(\boldsymbol{f}|\boldsymbol{g})\} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\}$

  with $\quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\|\boldsymbol{D}\boldsymbol{f}\|^2, \qquad \lambda = \frac{\sigma_\epsilon^2}{\sigma_f^2}$

- Advantage : characterization of the solution

$$\boldsymbol{f}|\boldsymbol{g} \sim \mathcal{N}(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{P}}) \text{ with } \widehat{\boldsymbol{f}} = \widehat{\boldsymbol{P}}\boldsymbol{H}'\boldsymbol{g}, \quad \widehat{\boldsymbol{P}} = \left(\boldsymbol{H}'\boldsymbol{H} + \lambda\boldsymbol{D}'\boldsymbol{D}\right)^{-1}$$

MAP estimation with other priors:

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \quad \text{with} \quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\Omega(\boldsymbol{f})$$

Separable priors:

- Gaussian: $p(f_j) \propto \exp\{-\alpha|f_j|^2\} \longrightarrow \Omega(\boldsymbol{f}) = \alpha\sum_j |f_j|^2$
- Gamma:
  $p(f_j) \propto f_j^\alpha \exp\{-\beta f_j\} \longrightarrow \Omega(\boldsymbol{f}) = \alpha\sum_j \ln f_j + \beta f_j$
- Beta:
  $p(f_j) \propto f_j^\alpha (1-f_j)^\beta \longrightarrow \Omega(\boldsymbol{f}) = \alpha\sum_j \ln f_j + \beta\sum_j \ln(1-f_j)$
- Generalized Gaussian: $p(f_j) \propto \exp\{-\alpha|f_j|^p\}$, $1 < p < 2 \longrightarrow \Omega(\boldsymbol{f}) = \alpha\sum_j |f_j|^p$,

Markovian models:

$$p(f_j|\boldsymbol{f}) \propto \exp\left\{-\alpha\sum_{i\in N_j}\phi(f_j, f_i)\right\} \longrightarrow \quad \Omega(\boldsymbol{f}) = \alpha\sum_j\sum_{i\in N_j}\phi(f_j, f_i),$$

MAP estimation with markovien priors:

$$\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\} \quad \text{with} \quad J(\boldsymbol{f}) = \|\boldsymbol{g} - \boldsymbol{H}\boldsymbol{f}\|^2 + \lambda\Omega(\boldsymbol{f})$$

$$\Omega(\boldsymbol{f}) = \sum_j \phi(f_j - f_{j-1})$$

with $\phi(t)$:

Convex functions:

$$|t|^\alpha, \ \sqrt{1+t^2} - 1, \ \log(\cosh(t)), \ \begin{cases} t^2 & |t| \leq T \\ 2T|t| - T^2 & |t| > T \end{cases}$$

or Non convex functions:

$$\log(1+t^2), \quad \frac{t^2}{1+t^2}, \quad \arctan(t^2), \quad \begin{cases} t^2 & |t| \leq T \\ T^2 & |t| > T \end{cases}$$

Main advantages of the Bayesian approach

- MAP = Regularization
- Posterior mean ? Marginal MAP ?
- More information in the posterior law than only its mode or its mean
- Meaning and tools for estimating hyper parameters
- Meaning and tools for model selection
- More specific and specialized priors, particularly through the hidden variables
- More computational tools:
  - Expectation-Maximization for computing the maximum likelihood parameters
  - MCMC for posterior exploration
  - Variational Bayes for analytical computation of the posterior marginals
  - ...

Blind Deconvolution: Bayesian approach

Deconvolution	Identification
$\boldsymbol{g} = \boldsymbol{H}\,\boldsymbol{f} + \boldsymbol{\epsilon}$	$\boldsymbol{g} = \boldsymbol{F}\,\boldsymbol{h} + \boldsymbol{\epsilon}$
$p(\boldsymbol{g}\vert\boldsymbol{f}) = \mathcal{N}(\boldsymbol{H}\boldsymbol{f}, \boldsymbol{\Sigma}_\epsilon = \sigma_\epsilon^2 \boldsymbol{I})$	$p(\boldsymbol{g}\vert\boldsymbol{h}) = \mathcal{N}(\boldsymbol{F}\boldsymbol{h}, \boldsymbol{\Sigma}_\epsilon = \sigma_\epsilon^2 \boldsymbol{I})$
$p(\boldsymbol{f}) = \mathcal{N}(0, \boldsymbol{\Sigma}_f = \sigma_f^2 (\boldsymbol{D}'_f \boldsymbol{D}_f)^{-1})$	$p(\boldsymbol{h}) = \mathcal{N}(0, \boldsymbol{\Sigma}_h = \sigma_h^2 (\boldsymbol{D}'_h \boldsymbol{D}_h)^{-1})$
$p(\boldsymbol{f}\vert\boldsymbol{g}) = \mathcal{N}(\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{\Sigma}}_f)$	$p(\boldsymbol{h}\vert\boldsymbol{g}) = \mathcal{N}(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{\Sigma}}_h)$
$\widehat{\boldsymbol{\Sigma}}_f = [\boldsymbol{H}'\boldsymbol{H} + \lambda_f \boldsymbol{D}'_f \boldsymbol{D}_f]^{-1}$	$\widehat{\boldsymbol{\Sigma}}_h = [\boldsymbol{F}'\boldsymbol{F} + \lambda_h \boldsymbol{D}'_h \boldsymbol{D}_h]^{-1}$
$\widehat{\boldsymbol{f}} = [\boldsymbol{H}'\boldsymbol{H} + \lambda_f \boldsymbol{D}'_f \boldsymbol{D}_f]^{-1} \boldsymbol{H}'\boldsymbol{g}$	$\widehat{\boldsymbol{h}} = [\boldsymbol{F}'\boldsymbol{F} + \lambda_h \boldsymbol{D}'_h \boldsymbol{D}_h]^{-1} \boldsymbol{F}'\boldsymbol{g}$

- Joint posterior law:

$$p(\boldsymbol{f}, \boldsymbol{h}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f}, \boldsymbol{h})\, p(\boldsymbol{f})\, p(\boldsymbol{h})$$

$$p(\boldsymbol{f}, \boldsymbol{h}|\boldsymbol{g}) \propto \exp\{-J(\boldsymbol{f}, \boldsymbol{h})\}$$

with

$$J(\boldsymbol{f}, \boldsymbol{h}) = \|\boldsymbol{g} - \boldsymbol{h}\boldsymbol{f}\|^2 + \lambda_f \|\boldsymbol{D}_f \boldsymbol{f}\|^2 + \lambda_h \|\boldsymbol{D}_h \boldsymbol{h}\|^2$$

- iterative algorithm

Blind Deconvolution: Bayesian Joint MAP criterion

- Joint posterior law:

$$p(\boldsymbol{f},\boldsymbol{h}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{h})\,p(\boldsymbol{f})\,p(h\boldsymbol{h})$$

$$p(\boldsymbol{f},\boldsymbol{h}|\boldsymbol{g}) \propto \exp\{-J(\boldsymbol{f},\boldsymbol{h})\}$$

with

$$J(\boldsymbol{f},\boldsymbol{h}) = \|\boldsymbol{g}-\boldsymbol{h}\boldsymbol{f}\|^2 + \lambda_f\|\boldsymbol{D}_f\boldsymbol{f}\|^2 + \lambda_h\|\boldsymbol{D}_h\boldsymbol{h}\|^2$$

- iterative algorithm

Deconvolution	Identification
$p(\boldsymbol{g}\|\boldsymbol{f},\boldsymbol{H}) = \mathcal{N}(\boldsymbol{H}\boldsymbol{f},\boldsymbol{\Sigma}_\epsilon)$	$p(\boldsymbol{g}\|\boldsymbol{h},\boldsymbol{F}) = \mathcal{N}(\boldsymbol{F}\boldsymbol{h},\boldsymbol{\Sigma}_\epsilon)$
$p(\boldsymbol{f}) = \mathcal{N}(0,\boldsymbol{\Sigma}_f)$	$p(\boldsymbol{h}) = \mathcal{N}(0,\boldsymbol{\Sigma}_h)$
$p(\boldsymbol{f}\|\boldsymbol{g},\boldsymbol{H}) = \mathcal{N}(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\Sigma}}_f)$	$p(\boldsymbol{h}\|\boldsymbol{g},\boldsymbol{F}) = \mathcal{N}(\widehat{\boldsymbol{h}},\widehat{\boldsymbol{\Sigma}}_h)$
$\widehat{\boldsymbol{\Sigma}}_f = [\boldsymbol{H}'\boldsymbol{H} + \lambda_f\boldsymbol{D}_f'\boldsymbol{D}_f]^{-1}$	$\widehat{\boldsymbol{\Sigma}}_h = [\boldsymbol{F}'\boldsymbol{F} + \lambda_h\boldsymbol{D}_h'\boldsymbol{D}_h]^{-1}$
$\widehat{\boldsymbol{f}} = [\boldsymbol{H}'\boldsymbol{H} + \lambda_f\boldsymbol{D}_f'\boldsymbol{D}_f]^{-1}\boldsymbol{H}'\boldsymbol{g}$	$\widehat{\boldsymbol{h}} = [\boldsymbol{F}'\boldsymbol{F} + \lambda_h\boldsymbol{D}_h'\boldsymbol{D}_h]^{-1}\boldsymbol{F}'\boldsymbol{g}$

Blind Deconvolution: Marginalization and EM algorithm

- Joint posterior law:
- Marginalization

$$p(\boldsymbol{f},\boldsymbol{h}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{h})\, p(\boldsymbol{f})\, p(h\boldsymbol{h})$$

$$p(\boldsymbol{h}|\boldsymbol{g}) = \int p(\boldsymbol{f},\boldsymbol{h}|\boldsymbol{g})\,\mathsf{d}\boldsymbol{f}$$

$$\widehat{\boldsymbol{h}} = \arg\max_{\boldsymbol{h}} \{p(\boldsymbol{h}|\boldsymbol{g})\} \longrightarrow \widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}} \left\{p(\boldsymbol{f}|\boldsymbol{g},\widehat{\boldsymbol{h}})\right\}$$

- Expression of $p(\boldsymbol{h}|\boldsymbol{g})$ and its maximization are complexes
- Expectation-Maximization Algorithm

$$\ln p(\boldsymbol{f},\boldsymbol{h}|\boldsymbol{g}) \propto J(\boldsymbol{f},\boldsymbol{h}) = \|\boldsymbol{g}-\boldsymbol{h}\boldsymbol{f}\|^2+\lambda_f\|\boldsymbol{D}_f\boldsymbol{f}\|^2+\lambda_h\|\boldsymbol{D}_h\boldsymbol{h}\|^2$$

  - Iterative algorithm
  - Expectation: Compute

$$Q(\boldsymbol{h},\boldsymbol{h}^{k-1}) = \mathsf{E}_{p(\boldsymbol{f},\boldsymbol{h}^{k-1}|\boldsymbol{g})}\{J(\boldsymbol{f},\boldsymbol{h})\} = \langle \ln p(\boldsymbol{f},\boldsymbol{h}|\boldsymbol{g})\rangle_{p(\boldsymbol{f},\boldsymbol{h}^{k-1}|\boldsymbol{g})}$$

  - Maximization:

$$\boldsymbol{h}^k = \arg\max_{\boldsymbol{h}} \left\{Q(\boldsymbol{h},\boldsymbol{h}^{k-1})\right\}$$

Blind Deconvolution: Variational Bayesian Approximation

- Joint posterior law:

$$p(\boldsymbol{f},\boldsymbol{h}|\boldsymbol{g}) \propto p(\boldsymbol{g}|\boldsymbol{f},\boldsymbol{h})\, p(\boldsymbol{f})\, p(h\boldsymbol{h})$$

- Approximation: $p(\boldsymbol{f},\boldsymbol{h}|\boldsymbol{g})$ by $q(\boldsymbol{f},\boldsymbol{h}|\boldsymbol{g}) = q_1(\boldsymbol{f})\, q_2(\boldsymbol{h})$
- Criterion of approximation: Kullback-Leiler

$$\mathsf{KL}(q|p) = \int q \,\ln \frac{q}{p} = \int q_1\, q_2 \,\ln \frac{q_1\, q_2}{p}$$

$$\begin{aligned}\mathsf{KL}(q_1\, q_2|p) &= \int q_1 \,\ln q_1 + \int q_2 \,\ln q_2 - \int q \ln p \\ &= -\mathcal{H}(q_1) - \mathcal{H}(q_2) + \langle - \ln p((\boldsymbol{f},\boldsymbol{h}|\boldsymbol{g})\rangle_q\end{aligned}$$

- When the expression of q_1 and q_2 are obtained, use them.

Variational Bayesian Approximation algorithm

- Kullback-Leibler criterion

$$\begin{aligned}\mathrm{KL}(q_1\, q_2|p) &= \int q_1 \ln q_1 + \int q_2 \ln q_2 + \int q \ln p \\ &= -\mathcal{H}(q_1) - \mathcal{H}(q_2) + \langle -\ln p((\boldsymbol{f}, \boldsymbol{h}|\boldsymbol{g})\rangle_q\end{aligned}$$

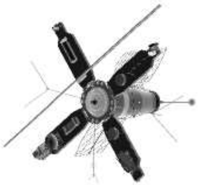
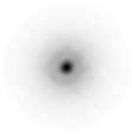
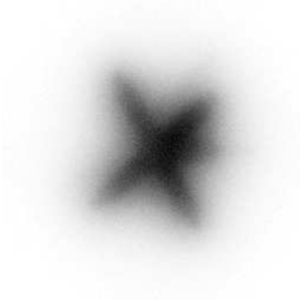
- Free energy

$$\mathcal{F}(q_1\, q_2) = -\langle \ln p((\boldsymbol{f}, \boldsymbol{h}|\boldsymbol{g})\rangle_{q_1 q_2}$$

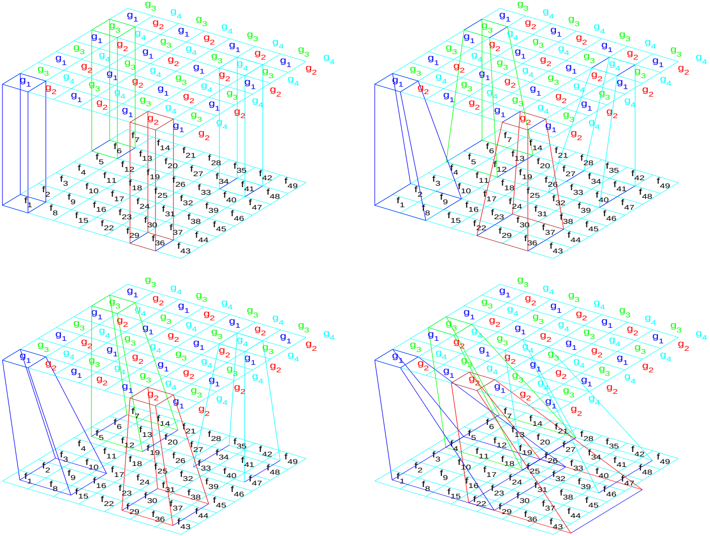
- Equivalence between optimization of $\mathrm{KL}(q_1\, q_2|p)$ and $\mathcal{F}(q_1\, q_2)$
- Alternate optimization:

$$\begin{aligned}\widehat{q}_1 &= \arg\min_{q_1}\{\mathrm{KL}(q_1\, q_2|p)\} = \arg\min_{q_1}\{\mathcal{F}(q_1\, q_2)\} \\ \widehat{q}_2 &= \arg\min_{q_2}\{\mathrm{KL}(q_1\, q_2|p)\} = \arg\min_{q_2}\{\mathcal{F}(q_1\, q_2)\}\end{aligned}$$

Deconvolution results


Forward modeling: from a HR image to LR images



$$g_k(\boldsymbol{r}) = [\mathcal{DBM}_k f](\boldsymbol{r})$$

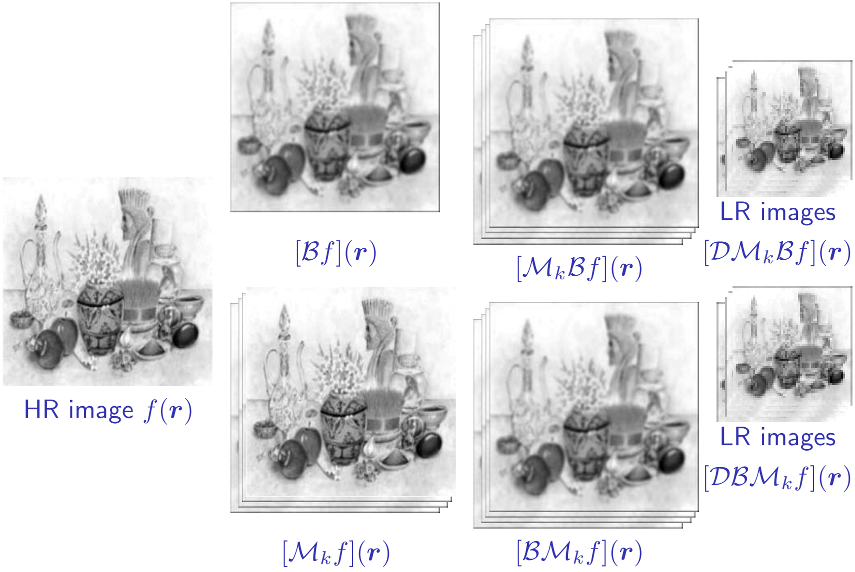
Modeling forward problems of superresolution

$$g_k(\boldsymbol{r}) = [\mathcal{H}_k f](\boldsymbol{r}) + \epsilon_k(\boldsymbol{r}) = [\mathcal{D}\mathcal{B}\mathcal{M}_k f](\boldsymbol{r}) + \epsilon_k(\boldsymbol{r})$$

- $\mathcal{B}$: Blurring effects (needs deconvolution)
- $\mathcal{M}_k$: Movement effects (needs registration)
- $\mathcal{D}$: Sub sampling effects (needs interpolation)
- Two models:

$$\begin{aligned} g_k(\boldsymbol{r}) &= [\mathcal{D}\mathcal{B}\mathcal{M}_k f](\boldsymbol{r}) + \epsilon_k(\boldsymbol{r}) \\ &= [\mathcal{D}\mathcal{M}_k\mathcal{B} f](\boldsymbol{r}) + \epsilon_k(\boldsymbol{r}) \end{aligned}$$

Two models for MISO SR problems



Classical methods

- Many classical methods are based on adjoint operators:

$$g_k(\boldsymbol{r}) = [\mathcal{D}\mathcal{B}\mathcal{M}_k f](\boldsymbol{r}) \longrightarrow f(\boldsymbol{r}) = \sum_k [\mathcal{M}'_k \mathcal{B}' \mathcal{D}' g_k](\boldsymbol{r})$$

- 3 basic operations:
  Interpolation, Registration and Image fusion
  - Interpolation: an ad hoc way to upsampling
  - Registration: compensation for movements
    Correlation based methods:

$$f_1(\boldsymbol{r}) \quad Movement \quad f_2(\boldsymbol{r}) = f_1(\boldsymbol{r}-\boldsymbol{d})$$

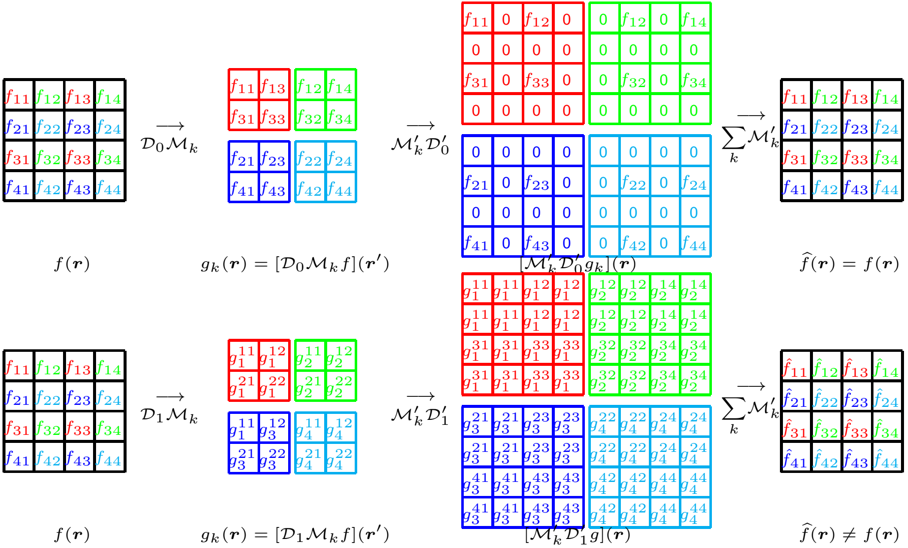
$$C(\boldsymbol{r}') = \int f_1(\boldsymbol{r}) f_2(\boldsymbol{r}-\boldsymbol{r}') \, \mathrm{d}\boldsymbol{r} = \delta(\boldsymbol{r}'-\boldsymbol{d})$$

    Fourier domain based methods:

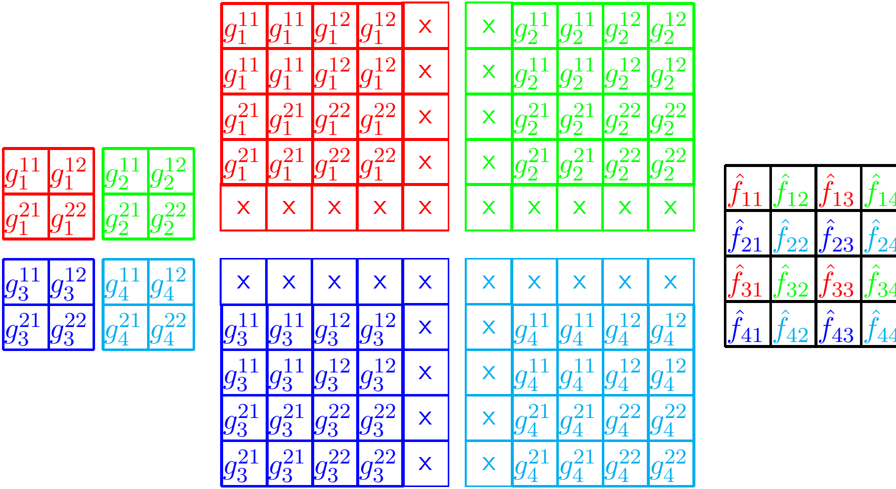
$$\begin{array}{ccc} f(\boldsymbol{r}) & \overset{FT}{\leftrightarrow} & F(\boldsymbol{\omega}) \\ f(\boldsymbol{r}-\boldsymbol{d}) & \leftrightarrow & \exp\{-j\boldsymbol{\omega}'\boldsymbol{d}\} \, F(\boldsymbol{\omega}) \end{array}$$

  - Image fusion: linear (mean) or non linear (median)

Classical methods: adjoint operators



Interpolation, HR registration and image fusion



Classical SR methods

Three main methods:

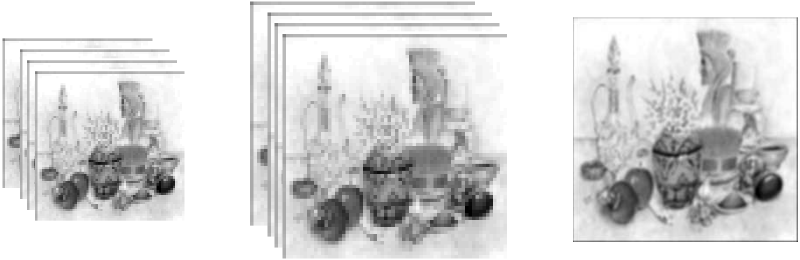
- $f(\boldsymbol{r}) = \sum_k \left[\mathcal{B}'\mathcal{D}'\mathcal{M}'_k g_k\right](\boldsymbol{r})$
  - Sub pixel LR registration
  - Interpolation to HR grid
  - Mean or Median image fusion

- $f(\boldsymbol{r}) = \sum_k \left[\mathcal{M}'_k\mathcal{B}'\mathcal{D}' g_k\right](\boldsymbol{r})$
  - Interpolation of all LR images to HR grid
  - HR registration
  - Mean or Median image fusion

- Iterative Backprojection methods

$$f^{(i+1)} = f^{(i)} + \alpha \sum_k [\mathcal{B}'\mathcal{D}'\mathcal{M}'_k] \left(g_k - \mathcal{M}_k\mathcal{D}\mathcal{B}f^{(i)}\right)$$

Classical SR methods



Unwrapping and interpolation $\longrightarrow$ Linear or Non Lineaire combination

Sub pixel registration + Interpolation + Mean or Median image fusion

Interpolation + HR registration + Mean or Median image fusion

General inversion methods

$$\begin{array}{lll} g_k(\boldsymbol{r}) & = [\mathcal{H}_k f](\boldsymbol{r}) + \epsilon_k(\boldsymbol{r}) & = [\mathcal{D}\mathcal{M}_k\mathcal{B}f](\boldsymbol{r}) + \epsilon_k(\boldsymbol{r}) \\ \boldsymbol{g}_k & = \boldsymbol{H}_k \boldsymbol{f} + \boldsymbol{\epsilon} & = \boldsymbol{D}\boldsymbol{M}_k\boldsymbol{B}\boldsymbol{f} + \boldsymbol{\epsilon} \end{array}$$

- Least squares (LS) methods: $\widehat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \{J(\boldsymbol{f})\}$

$$J(\boldsymbol{f}) = \sum_k \|\boldsymbol{g}_k - \boldsymbol{H}_k \boldsymbol{f}\|^2 = \sum_k \sum_{\boldsymbol{r}\in\mathcal{R}} |g_k(\boldsymbol{r}) - [\mathcal{H}_k f](\boldsymbol{r})|^2$$

- Iterative algorithms

$$\begin{array}{ll} \widehat{\boldsymbol{f}}^{(k+1)} & = \widehat{\boldsymbol{f}}^{(k)} - \alpha \nabla J(\widehat{\boldsymbol{f}}^{(k)}) \\ & = \widehat{\boldsymbol{f}}^{(k)} + 2\alpha \sum_k \boldsymbol{H}'_k(\boldsymbol{g}_k - \boldsymbol{H}_k \widehat{\boldsymbol{f}}^{(k)}) \end{array}$$

General inversion methods

- Regularization methods (Tikhonov)
$$J(\boldsymbol{f}) = \sum_k \|\boldsymbol{g}_k - \boldsymbol{H}_k \boldsymbol{f}\|^2 + \lambda \|\boldsymbol{D}\boldsymbol{f}\|^2$$
- Robust estimation (RE) [7, 8] [9, 7, 8, 10]
$$J(\boldsymbol{f}) = \sum_k \|\boldsymbol{g}_k - \boldsymbol{H}_k \boldsymbol{f}\|^{\beta_1} + \lambda \|\boldsymbol{D}\boldsymbol{f}\|^{\beta_2},\ 1 < \beta_1, \beta_2 \leq 2$$
- Bayesian MAP estimation methods [1, 2, 3, 4, 5, **?**].
$$J(\boldsymbol{f}) = -\ln p(\boldsymbol{f}|\underline{\boldsymbol{g}}) = -\ln p(\underline{\boldsymbol{g}}|\boldsymbol{f}) - \ln p(\boldsymbol{f}) + c$$
$$p(\underline{\boldsymbol{g}}|\boldsymbol{f}) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2} \sum_k \|\boldsymbol{g}_k - \boldsymbol{H}_k \boldsymbol{f}\|^2\right\}$$
and $p(\boldsymbol{f})$ is the prior law

General Bayesian inference

- Use forward and errors model to obtain the likelihood

$$\boldsymbol{g}_k = \boldsymbol{H}_k \boldsymbol{f} + \boldsymbol{\epsilon} = \boldsymbol{D}\boldsymbol{M}_k \boldsymbol{B}\boldsymbol{f} + \boldsymbol{\epsilon}$$

$$p(\underline{\boldsymbol{g}}|\boldsymbol{f}) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\sum_k \|\boldsymbol{g}_k - \boldsymbol{H}_k\boldsymbol{f}\|^2\right\}$$

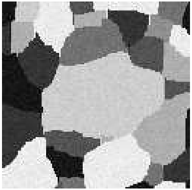
- Use prior knowledge to assign prior law $p(\boldsymbol{f})$
- Obtain the expresion of the posterior law

$$p(\boldsymbol{f}|\underline{\boldsymbol{g}}) = \frac{p(\underline{\boldsymbol{g}}|\boldsymbol{f})\, p(\boldsymbol{f})}{p(\underline{\boldsymbol{g}})}$$

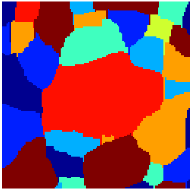
- Use it to make inference: MAP or PM

$$\widehat{\boldsymbol{f}} = \arg\max_{\boldsymbol{f}}\{p(\boldsymbol{f}|\underline{\boldsymbol{g}})\} \quad \text{or} \quad \widehat{\boldsymbol{f}} = \int \boldsymbol{f}\, p(\boldsymbol{f}|\underline{\boldsymbol{g}})\, \mathrm{d}\boldsymbol{f}$$

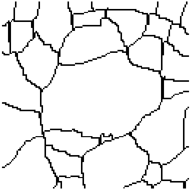
Hidden Markov model variables



$f(\boldsymbol{r})$



region labels $z(\boldsymbol{r})$



contours $q(\boldsymbol{r})$

$$\begin{array}{lcl} p(f(\boldsymbol{r})|z(\boldsymbol{r})=k) & = & \mathcal{N}(m_k, v_k^2), \quad P(z(\boldsymbol{r})=k)=\alpha_k \\ p(f(\boldsymbol{r})) & = & \sum_{k=1}^K \alpha_k \mathcal{N}(m_k, v_k^2) \end{array}$$

$$\mathcal{R}_k = \cup_l \mathcal{R}_{kl} \text{ with } \cap_l \mathcal{R}_{kl} = 0, \; \cap_k \mathcal{R}_k = 0, \; \cup_k \mathcal{R}_k = \mathcal{R}$$

$$p(z(\boldsymbol{r})|z(\boldsymbol{r}'), \boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})) \propto \exp\left\{ \gamma \sum_{\boldsymbol{r}' \in \mathcal{V}(\boldsymbol{r})} \delta(z(\boldsymbol{r}) - z(\boldsymbol{r}')) \right\}$$

Expression of the posterior law

- Likelihood: $p(\underline{\boldsymbol{g}}|\boldsymbol{f},\sigma_\epsilon^2) \propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\sum_k \|\boldsymbol{g}_k - \boldsymbol{H}_k\boldsymbol{f}\right\}$
- Prior laws:

$$p(\boldsymbol{f}|\boldsymbol{z},\{m_k,v_k\}) \propto \prod_{k=1}^{K}\prod_{\boldsymbol{r}\in\mathcal{R}_k} \exp\left\{\frac{-1}{2v_k^2}(f(\boldsymbol{r})-m_k)^2\right\}$$

$$p(\boldsymbol{z}|\gamma) \propto \exp\left\{\gamma\sum_{\boldsymbol{r}\in\mathcal{R}}\sum_{\boldsymbol{r}'\in\mathcal{V}(\boldsymbol{r})}\delta(z(\boldsymbol{r})-z(\boldsymbol{r}'))\right\}$$

$$p(\boldsymbol{\theta}) = p(\sigma_\epsilon^2)\prod_k p(m_k)\,p(v_k)\,p(\gamma)$$

- Joint Posterior law of all the unknowns:

$$p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\underline{\boldsymbol{g}}) \quad\propto\quad p(\underline{\boldsymbol{g}}|\boldsymbol{f},\sigma_\epsilon^2)\,p(\boldsymbol{f}|\boldsymbol{z},\{m_k,v_k\})\,p(\boldsymbol{z}|\gamma)\,p(\boldsymbol{\theta})$$

Bayesian computation

$$p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\underline{\boldsymbol{g}}) \propto p(\underline{\boldsymbol{g}}|\boldsymbol{f},\sigma_\epsilon^2)\; p(\boldsymbol{f}|\boldsymbol{z},\{m_k,v_k\})\; p(\boldsymbol{z}|\gamma)\; p(\boldsymbol{\theta})$$

- Joint MAP : (Optimization)

$$(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}}) = \arg\max_{(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta})} \{p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta}|\underline{\boldsymbol{g}})\}$$

- Posterior means: (Integration)

$$\widehat{\boldsymbol{f}} = \mathsf{E}\left\{\boldsymbol{f}|\underline{\boldsymbol{g}}\right\}, \quad \boldsymbol{\theta} = \mathsf{E}\left\{\boldsymbol{\theta}|\underline{\boldsymbol{g}}\right\}, \quad \widehat{\boldsymbol{z}} = \mathsf{E}\left\{\boldsymbol{z}|\underline{\boldsymbol{g}})\right\}$$

- General iterative algorithms:

$$\begin{array}{lcll} \widehat{\boldsymbol{f}} & \sim & p(\boldsymbol{f}|\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}},\underline{\boldsymbol{g}}) & \text{Estimation} \\ \widehat{\boldsymbol{z}} & \sim & p(\boldsymbol{z}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{\theta}},\underline{\boldsymbol{g}}) & \text{Segmentation} \\ \widehat{\boldsymbol{\theta}} & \sim & p(\boldsymbol{\theta}|\widehat{\boldsymbol{f}},\widehat{\boldsymbol{z}},\underline{\boldsymbol{g}}) & \text{Hyperparameters} \end{array}$$

Bayesian SR

- In real SR problems, we have also to estimate the PSF $\boldsymbol{h}$ and the movement or registration parameters $\boldsymbol{d}_k$

$$p(\boldsymbol{f},\boldsymbol{z},\boldsymbol{\theta},\boldsymbol{h},\boldsymbol{d}_k|\underline{\boldsymbol{g}}) \propto \begin{array}{l} p(\underline{\boldsymbol{g}}|\boldsymbol{f},\sigma_\epsilon^2)\, p(\boldsymbol{f}|\boldsymbol{z},\{m_k,v_k\}) \\ p(\boldsymbol{z}|\gamma)\, p(\boldsymbol{\theta})\, p(\boldsymbol{h})\, \prod_k p(\boldsymbol{d}_k) \end{array}$$

- General iterative algorithms:
  - Update PSF $\boldsymbol{h}$
  - Update registration parameters $\boldsymbol{d}_k$
  - Update segmentation $\boldsymbol{z}$ and contours $\boldsymbol{q}$
  - Update registration parameters $\boldsymbol{\theta}$
  - Update the HR image $\boldsymbol{f}$

A first algorithm

- Initialization:
  1. Estimate the sub-pixel translational movements $\boldsymbol{d}_k$ between the LR images $\boldsymbol{g}_k(\boldsymbol{r})$;
  2. Estimate a first HR image $\widehat{\boldsymbol{f}}(\boldsymbol{r})$ based on LS or quadratic regularization

- Iterations:
  1. Estimate a segmentation $\widehat{\boldsymbol{z}}(\boldsymbol{r})$ for the HR image $\widehat{\boldsymbol{f}}(\boldsymbol{r})$ based on the Potts Markov modeling;
  2. Estimate the parameters $\widehat{\boldsymbol{\theta}}$ of Gaussian mixtures;
  3. Update the HR image using:

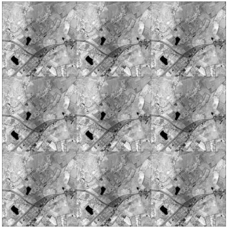
$$\ln p(\boldsymbol{f}|\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\theta}},\underline{\boldsymbol{g}}) = \sum_k \|\boldsymbol{g}_k - \boldsymbol{H}_k \boldsymbol{f}\|^2 + \sum_k \sum_{\boldsymbol{r}\in\mathcal{R}_k} \left(\frac{f(\boldsymbol{r}) - m_k}{v_k}\right)^2$$

New algorithm

- Initialization:
  1. Estimate a first HR image $\widehat{\boldsymbol{f}}(\boldsymbol{r})$ by interpolating the first LR image;
- Iterations:
  1. Estimate the translational movements $\boldsymbol{d}_k$ between the HR image $\widehat{\boldsymbol{f}}(\boldsymbol{r})$ and newly entered LR images $\boldsymbol{g}_k(\boldsymbol{r})$ which is interpolated to the HR dimensions.
  2. Estimate the blurring PSF $\boldsymbol{h}$.
  3. Estimate a segmentation $\widehat{\boldsymbol{z}}(\boldsymbol{r})$ for the HR image
  4. Estimate the parameters $\widehat{\boldsymbol{\theta}}$ of Gaussian mixtures
  5. Update the HR image as before

Results

Original HR $\boldsymbol{f}_0$



LR images with $k = 3$

One of the LR images

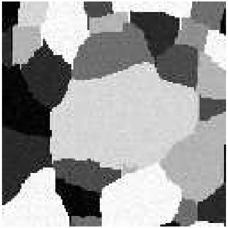
Robust regularization

$$df = \frac{\|\widehat{f} - f_0\|^2}{\|f_0\|^2} = 9\%$$

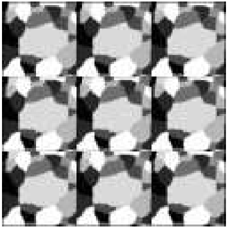
Proposed method

$$df = \frac{\|\widehat{f} - f_0\|^2}{\|f_0\|^2} = 7\%$$

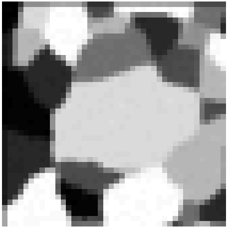
Results



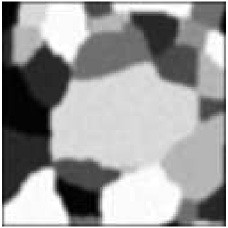
Original HR $\boldsymbol{f}_0$



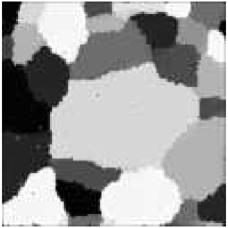
LR images with $k = 3$



One of the LR images



Robust regularization
$df = \frac{\|\widehat{f} - f_0\|^2}{\|f_0\|^2} = 4.9\%$



Proposed method
$df = \frac{\|\widehat{f} - f_0\|^2}{\|f_0\|^2} = 2.8\%$

Conclusions

- The Bayesian approach is an appropriate approach for any inverse problem, and so, for SR problem.
- In this approach, it is possible to account for any prior knowledge.
- The uncertainties in each step are transmitted to the following steps in a natural way through the probability laws.
- Obtained methods give more satisfaction if the forward and prior models are more appropriate.
- In general, the computational costs are higher than classical methods, but non really so much.

Challenges

- Forward modeling: Translation, Rotation, Zooming and other projective models for registration step has to be accounted for.
- Prior modeling: Accounting for textures in each region
- More efficient and robust movement, or more generally, registration parameters estimation algorithms
- More efficient PSF estimation algorithms
- More efficient optimization algorithms, and more generally, Bayesian computation methods (MCMC, Variational methods)

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Questions and comments ?