COPULA AND TOMOGRAPHY

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Keywords: Copula, Tomography, Joint and marginal distributions, Image reconstruction, Backprojection, Archimedian, Gaussian and Cubic copulas

Abstract: An important problem in statistics is determining a joint probability distribution from its marginals. In 2D case, the marginal probability density functions $f_1(x)$ and $f_2(y)$ are related to their joint distribution $f(x,y)$ via the horizontal and vertical line integrals. So, the problem of determining $f(x,y)$ from $f_1(x)$ and $f_2(y)$ is an ill-posed inverse problem. In statistics the notion of *copula* is exactly introduced to obtain a solution to this problem. Interestingly, this is also a problem encountered in X ray tomography image reconstruction where $f(x,y)$ is an image representing the distribution of the material density and $f_1(x)$ and $f_2(y)$ are the horizontal and vertical line integrals. In this paper, we try to link the notion of copula to X ray Computed Tomography (CT) and to see if we can use the methods used in each domain to the other one.

1 Introduction

The word *copula* originates from the Latin meaning *link, chain, union*. In statistical literature, according to the seminal result in the copula's theory stated by Abe Sklar (Sklar, 1959) in 1959; A copula is a function that connects a multivariate distribution function to its given univariate marginal distributions. There is an increasing interest concerning copulas, widely used in Financial Mathematics (kallenberg, 2008), in modelling of Environmental Data (Joe, 1994). Recently, in Computational Biology, copulas are used for the reconstruction of accurate cellular networks (JM et al., 2008). Copula appeared to be a new powerful tool to model the structure of dependence. Copulas are useful for constructing joint distributions, particularly with nonnormal random variables (JM et al., 2008; Yan, 2007; Genest and Favre, 2007; Mikosch, 2006; Genest and Rémillard, 2006; Zhang et al., 2006; Kolesárová et al., 2006; Durrleman et al., 2000).

In 2D case, the marginal probability density functions $f_1(x)$ and $f_2(y)$ are related to their joint probability density function $f(x,y)$ via the horizontal and vertical line integrals:

$$f_1(x) = \int f(x,y)\,\mathrm{d}y \quad (1)$$

$$f_2(y) = \int f(x,y)\,\mathrm{d}x \quad (2)$$

Given $f(x,y)$ computing $f_1(x)$ and $f_2(y)$ is a well-posed (forward) problem. The problem of determining $f(x,y)$ from $f_1(x)$ and $f_2(y)$ is an ill-posed (inverse) problem (Hadamard, 1902). As we will see later all functions in the form of

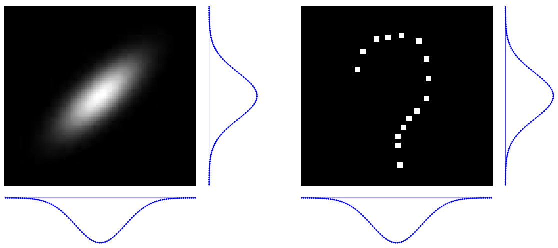
$$f(x,y) = f_1(x)\, f_2(y)\, c(x,y) \quad (3)$$

where $c(x,y)$ is any *copula* density function, is a solution of this problem. Later in detail a copula $c(x,y)$ will be a function such that its marginals are uniform and thus we have

$$\int f(x,y)\,\mathrm{d}y = \int [f_1(x)\, f_2(y)\, c(x,y)]\,\mathrm{d}y = f_1(x) \quad (4)$$

$$\int f(x,y)\,\mathrm{d}x = \int [f_1(x)\, f_2(y)\, c(x,y)]\,\mathrm{d}x = f_2(y) \quad (5)$$

In 1917, Johann Radon introduced the Radon transform (Radon, 1917) which is used in 1963 by Allan MacLeod Cormack for application in the con-



Forward problem:
Given $f(u,v)$ compute
$f_1(u)$ and $f_2(v)$

Inverse problem:
Given $f_1(u)$ and $f_2(v)$
determine $f(u,v)$

Figure 1: Forward and inverse problems

text of tomographic image reconstruction. He proposed to reconstruct the spatial variation of the material density of the body from X-Ray radiographies for different directions. He implemented this method and made a prototype CT scanner (Cormack, 1963). Independently, Godfrey Newbold Hounsfield derived an algorithm and built the first medical CT scanner. This was a great achievement for the twentieth century, because one could see inside of an object without opening it up. Cormack and Hounsfield won the Nobel Prize of Medicine in 1979.

Interestingly, if we represent by $f(x,y)$ the spatial distribution of the material density in a section of the body, a very simple model to relate a line of the radiography image $p_\theta(r)$ at direction θ to $f(x,y)$ is given by the Radon transform:

$$\begin{aligned} p_\theta(r) &= \int_{L_{r,\theta}} f(x,y)\,\mathrm{d}l \\ &= \iint_{\mathcal{R}^2} f(x,y)\delta(r - x\cos\theta - y\sin\theta)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

The mathematical problem is then determining the multivariate function $f(x,y)$ from its line integrals $p_\theta(r)$. Radon has shown that this problem has a unique solution if we know $p_\theta(r)$ for all $\theta \in [0,\pi]$ and all $r \in \mathcal{R}$ and can be computed by

$$f(x,y) = \frac{1}{2\pi}\int_0^\pi \int_0^\infty \frac{\frac{\partial p_\theta(r)}{\partial r}}{r - x\cos\theta - y\sin\theta}\,\mathrm{d}r\,\mathrm{d}\theta \quad (6)$$

However, if the number of projections is limited, then the problem is ill-posed and the problem has an infinite number of solutions.

If now, we consider only two projections: horizontal $\theta = 0$ and vertical $\theta = \pi/2$, we see easily the link between these two problems. The main objective of this paper is to show in more details these relations.

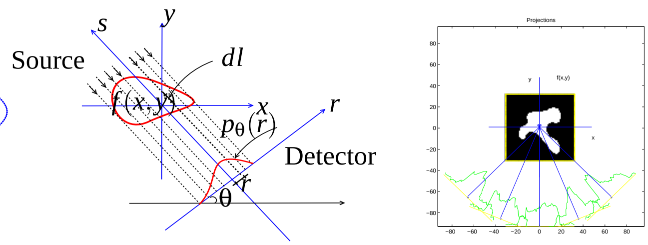


Figure 2: X ray Computed Tomography: forward and inverse problems in 2D parallel geometry.

The rest of this paper is organized as follows: In section 2, we present a summary of all the necessary definitions and properties of copulas and highlight methods to generate a copula. In section 3, we present a summary of the main tomographic image reconstruction methods based on the Radon inversion formula. In section 4, we will be in the heart of the new material of this paper which is the link and relations between the notions of these two previous sections. Finally, in section 5, we show some preliminary results from our Copula-Tomography Matlab package.

2 Copula

In this section, we give a few definitions and properties of copulas that we need in the rest of the paper. First, we note by $F(u,v)$ a bivariate cumulative distribution function (cdf), by $f(u,v)$ its bivariate probability density function (pdf), by $F_1(u)$, $F_2(v)$ its marginal cdf's and $f_1(u)$, $f_2(v)$ their corresponding pdf's with their classical relations:

$$F_1(u) = \int_{-\infty}^{u} f_1(x)\,\mathrm{d}x = F(u,\infty), \quad (7)$$

$$F_2(v) = \int_{-\infty}^{v} f_2(y)\,\mathrm{d}y = F(\infty,v), \quad (8)$$

$$F(u,v) = \int_{-\infty}^{u}\int_{-\infty}^{v} f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \quad (9)$$

$$f_1(u) = \frac{\partial F_1(u)}{\partial u} = \int f(u,v)\,\mathrm{d}v, \quad (10)$$

$$f_2(v) = \frac{\partial F_2(v)}{\partial v} = \int f(u,v)\,\mathrm{d}u, \quad (11)$$

$$f(u,v) = \frac{\partial^2 F(u,v)}{\partial u\,\partial v}. \quad (12)$$

Definition 1 *Bivariate Copula: A bivariate copula, or shortly a copula is a function from* $[0,1]^2$ *to* $[0,1]$ *with the following properties:*

- $\forall u,v \in [0,1], \quad C(u,0) = 0 = C(0,v)$;

- $\forall u,v \in [0,1], \quad C(u,1)=u \quad and \quad C(1,v)=v;$
- $\forall u_1,u_2,v_1,v_2 \in [0,1]$ *such that* $u_1 \le u_2$ *and* $v_1 \le v_2$,
  $V_C([u_1,u_2]\times[v_1,v_2]) = C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \ge 0.$

Theorem 1 *Sklar's Theorem: Let F be a two-dimensional distribution function with marginal distributions functions F_1 and F_2. Then there* ***exists*** *a copula C such that:*

$$F(u,v) = C(F_1(x_1),F_2(x_2)). \tag{13}$$

Conversely*, for any univariate distribution functions F_1 and F_2 and any copula C, the function F is a two-dimensional distribution function with marginals F_1 and F_2, given by (13). Furthermore, if the marginal functions are continuous, then the copula C is* ***unique****, and is given by*

$$C(u,v) = F(F_1^{-1}(u),F_2^{-1}(v)). \tag{14}$$

Definition 2 *Copula Density: From (12) and differentiating (14) gives the density of a copula*

$$c(u,v) = \frac{\partial^2 C}{\partial u \partial v} = \frac{f\left(F_1^{-1}(u),F_2^{-1}(v)\right)}{f_1\left(F_1^{-1}(u)\right)\ f_2\left(F_2^{-1}(v)\right)}, \tag{15}$$

and thus

$$f(x,y) = f_1(x)\, f_2(y)\, c(x,y) \tag{16}$$

Usual copulas:
The **product copula** $\Pi(u,v)$ (or independent copula) is the simplest copula, has the form

$$\Pi(u,v) = u\,v \quad (u,v) \in [0,1]^2, \tag{17}$$

corresponds to independence.
The **Fréchet-Hoeffding upper bound copula** $M(u,v)$ (or comonotonicity copula) is :

$$M(u,v) = \min(u,v) \quad (u,v) \in [0,1]^2. \tag{18}$$

The **Fréchet-Hoeffding lower bound** $W(u,v)$ (or countermonotonicity copula) is:

$$W(u,v) = \max\{u+v-1,0\} \quad (u,v) \in [0,1]^2. \tag{19}$$

Property 1 *Any copula $C(u,v)$, satisfies the inequality called the Fréchet-Hoeffding bound inequality*

$$W(u,v) \le C(u,v) \le M(u,v). \tag{20}$$

Generating Copulas by the Inversion Method: A straight forward method is based directly on Sklar's theorem. Given $F(x_1,x_2)$ the joint cdf of two variables X_1 and X_2 and $F_1(x_1)$ and $F_2(x_2)$ their marginal cdf's, all assumed to be continuous. The corresponding copula can be constructed using the unique inverse transformations (Quantile transform) $X_1 = F_1^{-1}(u)$, $X_2 = F_2^{-1}(v)$, where U and V are uniformly distributed on $[0,1]$:

$$C(u,v) = F(F_1^{-1}(u),F_2^{-1}(v)), \tag{21}$$

where u,v are uniform on $[0,1]$.

Archimedean Copulas: The Archimedean copulas form an important class of copulas ((Nelsen, 1999) page 89) which generalise the usual copulas.

Theorem 2 *Let φ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1)=0$, and let $\varphi^{[-1]}$ be the pseudo-inverse of φ. Let C be the function from $[0,1]^2$ to $[0,1]$ given by*

$$C(u_1,u_2) = \varphi^{[-1]}\left(\varphi(u_1)+\varphi(u_2)\right). \tag{22}$$

Then C is a copula if and only if φ is convex.

Archimedean copulas are in the form (22) and the function φ is called the generator of the copula. φ is a strict generator if $\varphi(0)=\infty$, then $\varphi^{[-1]} = \varphi^{-1}$ and

$$C(u,v) = \varphi^{-1}\left(\varphi(u)+\varphi(v)\right). \tag{23}$$

Property 2 *The following algebraic properties are satisfied by any Archimedean copula C, those properties distinguish this class of copula from all other copula.*

1. *$C(u_1,u_2) = C(u_2,u_1)$ meaning that C is symmetric $\forall u_1,u_2 \in [0,1]$;*
2. *C is associative $\forall u_1,u_2,u_3 \in [0,1]$ i.e. $C(C(u_1,u_2),u_3) = C(u_1,C(u_2,u_3))$;*
3. *If $a>0$ is any constant then $a\varphi$ is again a generator of C.*

Theorem 3 *Let C be an Archimedean copula with generator φ in Ω. Then for almost all u_1 and u_2 in $[0,1]$,*

$$\varphi'(u_1)\frac{\partial C(u_1,u_2)}{\partial u_2} = \varphi'(u_2)\frac{\partial C(u_1,u_2)}{\partial u_1}. \tag{24}$$

Definition 3 *If $F(x_1,x_2,\cdots,x_n)$, and $F_i(x_i)$ denoted respectively the multivariate distribution and its marginal functions, one particularly simple form of a $n-$dimensional Archimedean is*

$$F(x_1,x_2,\cdots,x_n) = \varphi^{-1}\left(\sum_{i=1}^{n}\varphi(F_i(x_i))\right), \tag{25}$$

where φ is the generator function such that $\varphi(1)=0$, $\varphi(0)=\infty$; and satisfies the convexity properties $\varphi'(x)<0$, $\varphi''(x)>0$.

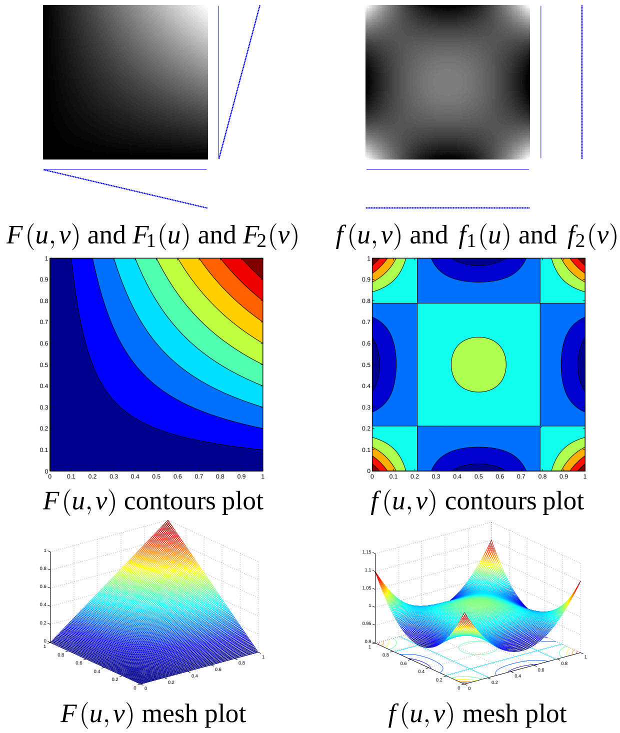


Figure 3: Cubic copula: different presentations

Property 3 *One easy way to compute the bivariate copula density function $c(u_1,u_2)$ of the copula $C(u_1,u_2)$, using the generator function φ under some conditions is given by:*

$$c(u_1,u_2) = -\frac{\varphi''(C(u_1,u_2))\varphi'(u_1)\varphi'(u_2)}{\left[\varphi'(C(u_1,u_2))\right]^3}. \tag{26}$$

Property 4 *Other rigorous mathematics way to define the Archimedean copula is related to the Laplace transform (for details and beauty of this method, we refer to (Marshall and Olkin, 1988)).*
Let Λ be a distribution function with support $\mathbb{R}_+$ and φ its Laplace transform,

$$\varphi(t) = \int_0^\infty \exp(-tx)\,\Lambda(dx), \tag{27}$$

φ is strictly nondecreasing function, $\varphi(0) = 1, \varphi(+\infty) = 0$, then the following relation define a copula

$$C(u_1,\ldots,u_n) = \varphi\left(\sum_{i=1}^{n} \varphi_i^{-1}(u_i)\right). \tag{28}$$

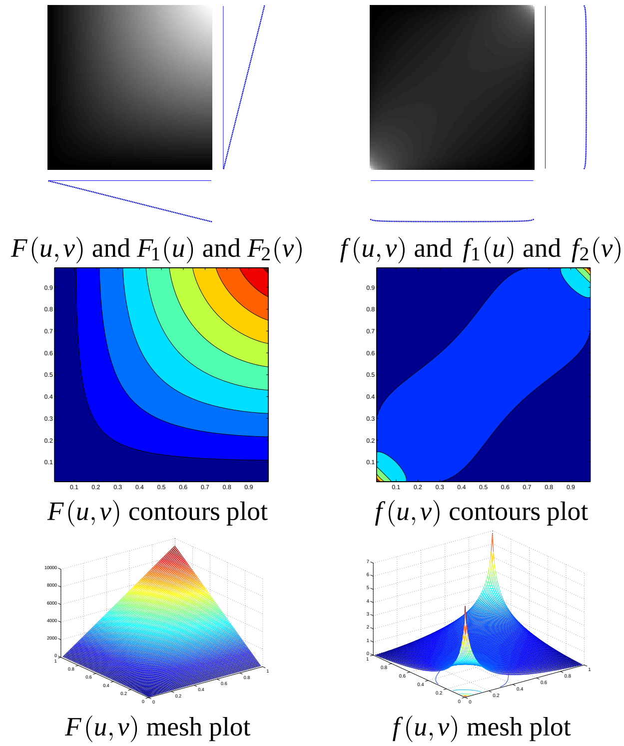


Figure 4: A Gaussian copula: different presentations

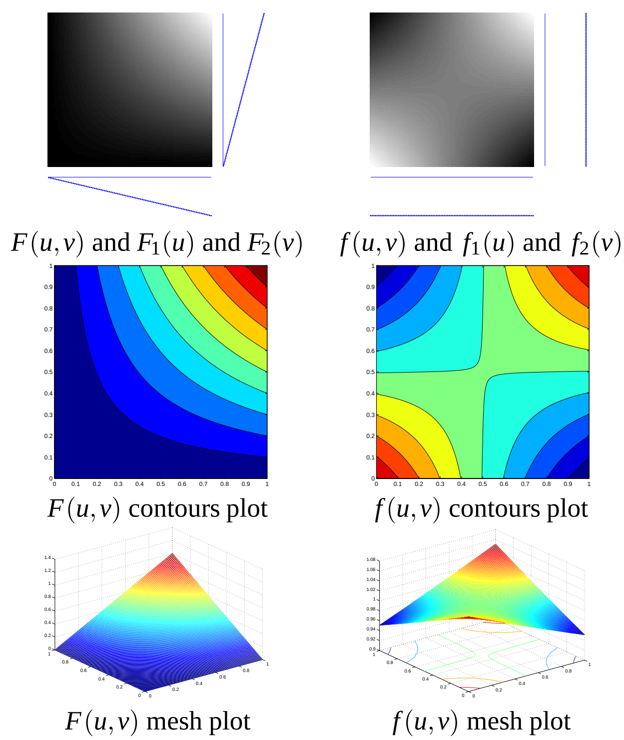


Figure 5: Franck copula: different presentations

3 Tomography

In X ray CT, the Radon Transform (RT) and its inverse:

$$
\begin{aligned}
p(r,\theta) &= \iint_{\mathcal{R}^2} f(x,y)\,\delta(r - x\cos\theta - y\sin\theta)\ \mathrm{d}x\ \mathrm{d}y \\
f(x,y) &= \frac{1}{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\frac{\partial p(r,\theta)}{\partial r}}{r - x\cos\theta - y\sin\theta}\ \mathrm{d}r\ \mathrm{d}\theta
\end{aligned}
$$

are the main relations. Decomposing the inverse RT in the following parts:

$$
\begin{aligned}
&\mathcal{D}: \quad \overline{p}_\theta(r) = \frac{\partial p(r,\theta)}{\partial r} \\
&\mathcal{H}: \quad \widetilde{\overline{p}}(r',\theta) = \frac{1}{\pi}\int_0^{\infty} \frac{\overline{p}(r,\theta)}{(r - r')}\ \mathrm{d}r \\
&\mathcal{B}: \quad f(x,y) = \frac{1}{2\pi}\int_0^{\pi} \widetilde{\overline{p}}(x\cos\theta + y\sin\theta, \theta)\ \mathrm{d}\theta
\end{aligned}
$$

and using the properties of the FT of the derivation and the relations between HT and FT, we obtain easily the following relations:

$$
f(x,y) = \mathcal{B}\,\mathcal{H}\,\mathcal{D}\,g(r,\theta) = \mathcal{B}\,\mathcal{F}_1^{-1}\,|\Omega|\,\mathcal{F}_1\,g(r,\theta) \tag{29}
$$

and the following classical method of filtered back-projection commonly used in X ray CT:

$$
\xrightarrow{g(r,\theta)} \boxed{\begin{matrix}\text{FT}\\ \mathcal{F}_1\end{matrix}} \longrightarrow \boxed{\begin{matrix}\text{Filter}\\ |\Omega|\end{matrix}} \longrightarrow \boxed{\begin{matrix}\text{IFT}\\ \mathcal{F}_1^{-1}\end{matrix}} \xrightarrow{g_1(r,\theta)} \boxed{\begin{matrix}\text{Backproj.}\\ \mathcal{B}\end{matrix}} \xrightarrow{f(x,y)}
$$

Also, if we define

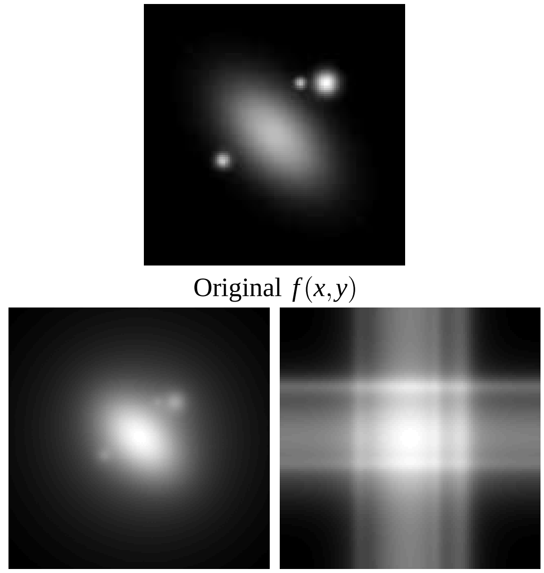
$$
b(x,y) = \frac{1}{2\pi}\int_0^{\pi} p(x\cos\theta + y\sin\theta, \theta)\ \mathrm{d}\theta \tag{30}
$$

then, it is shown that

$$
b(x,y) = f(x,y) * h(x,y) \tag{31}
$$

where $*$ stands for a 2D convolution and $h(x,y) = 1/\sqrt{x^2+y^2} = (x^2+y^2)^{-1/2}$.

In X-ray CT, if we have a great number of projections uniformly distributed over the $[0,\pi]$ angles, the filtered backprojection (FBP) image obtained by (29) or even the simple backprojection (BP) image by (30) are good solutions to the inverse CT problem as it is shown on the Figure 6 a). But, when we have only 2 projections, the FBP or BP images are not so good solutions as it is shown on the Figure 6 b).



a) BP with 128 projections b) BP with only 2 projections

Figure 6: BP and FBP methods with a great number of projections and with only two projections.

4 Link between Copula and Tomography

Now, let consider the particular case where we have only two projections $\theta = 0$ and $\theta = \pi/2$. Then

$$
\begin{aligned}
p_0(r) &= \iint f(x,y)\delta(r-x)\ \mathrm{d}x\ \mathrm{d}y = \int f(r,y)\ \mathrm{d}y \\
p_{\pi/2}(r) &= \iint f(x,y)\delta(r-y)\ \mathrm{d}x\ \mathrm{d}y = \int f(x,r)\ \mathrm{d}x
\end{aligned}
$$

and if we let note $f_1 = p_0$ and $f_2 = p_{\pi/2}$ we can deduce the following new methods for the inverse problem of determining $f(x,y)$ from $f_1(x)$ and $f_2(x)$:

Backprojection:

$$
f(x,y) = \frac{1}{2}(f_1(x) + f_2(y)). \tag{32}
$$

Filtered Backprojection:

$$
f(x,y) = \frac{1}{2}\left(\int \frac{\frac{\partial f_1}{\partial x}(x')}{x'-x}\ \mathrm{d}x' + \int \frac{\frac{\partial f_2}{\partial y}(y')}{y'-y}\ \mathrm{d}y'\right) \tag{33}
$$

which can also be implemented if the Fourier domaine.

$$
\begin{aligned}
f(x,y) = &\ \frac{1}{2}\int e^{+jux}|u|\left(\int e^{-jux} f_1(x)\ \mathrm{d}x\right)\ \mathrm{d}u \\
&+\frac{1}{2}\int e^{+jvy}|v|\left(\int e^{-jvy} f_2(y)\ \mathrm{d}y\right)\ \mathrm{d}v.
\end{aligned}
$$

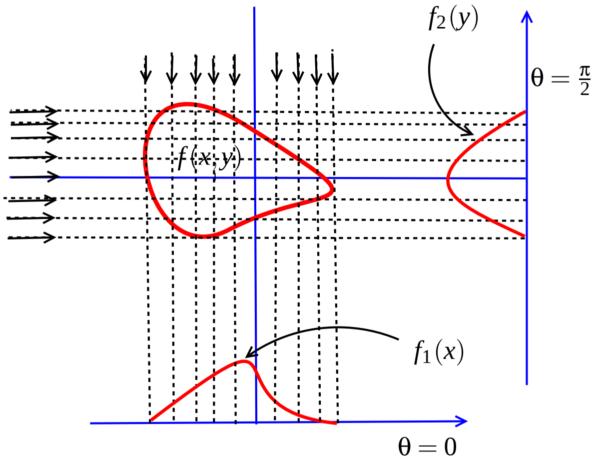


Figure 7: Link between Copulas and X ray tomography with only 2 projections.

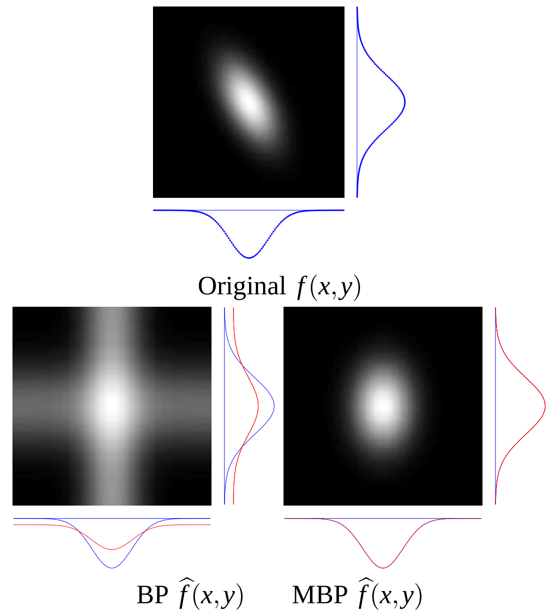


Figure 8: A comparison between BP and MBP with 2 projections. MBP image is better than BP image because it satisfies exactly the marginals.

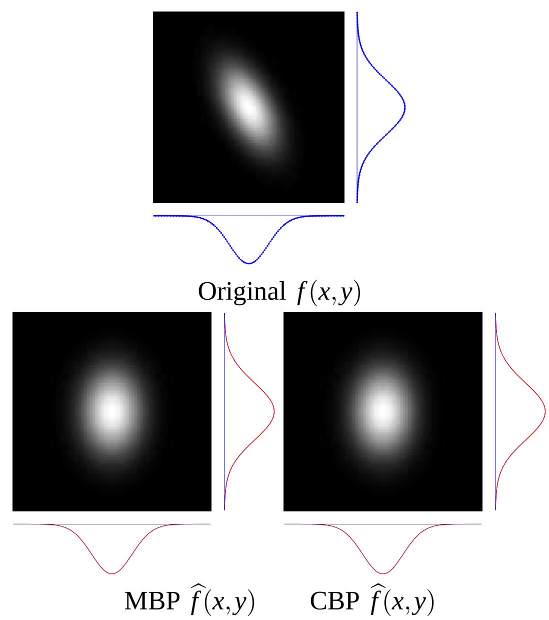


Figure 9: A comparison between MBP and CBP.

5 How to use Copula in Tomography

The definition and the notion of copula give us the possibility to propose a new X ray CT method. Let first consider the case of two projections. In this case, immediately, we can propose a first use which corresponds to the case of independent copula. We call this method *Multiplicative Backprojection (MBP).*

MBP:

$$f(x,y) = f_1(x)\, f_2(y) \tag{34}$$

This name comes naturally if we compare the two equations (32) and (34). In Figure (8) we see a comparisons of BP and MBP. As we can see, at least the image obtained by MBP is better than the one obtained by BP and it satisfies exactly the marginals.

We can do better if we used another copula than the independent copula by proposing the following method that we call *Copula Backprojection (CBP).*

CBP:

$$f(x,y) = f_1(x)\, f_2(y)\, c(x,y) \tag{35}$$

where $c(x,y)$ is a parametrized copula.

Let now, consider more complex images as it is shown in Figure (10). As we can see, even if the MBP images are better than BP images and the marginals of MBP are fitted exactly, these results are not really satisfactory. This is due to the fact that with only 2 projections we cannot reconstruct complex images and we need more projections.

As we can see with only two projections, there is not any hope to reconstruct a complexe shape object. We need more projections.

We had extended this idea to the general case which can be described as follows: In practice, we also need to normalize each projection in such a way that they can be assimilated to a pdf.

General MBP:

- Normalize each projection in such a way to satisfy $p_\theta(r) \geq 0$ and $\int p_\theta(r)\, \mathrm{d}r = 1$.

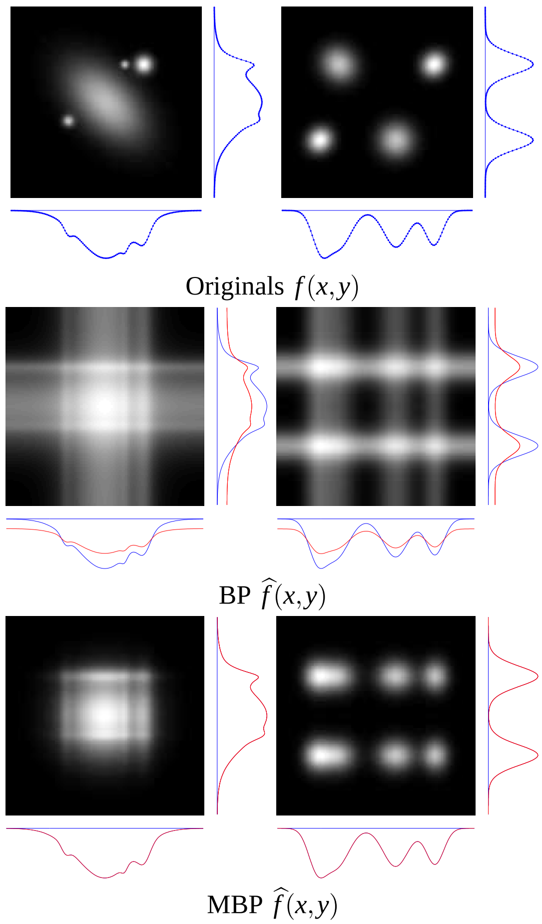


Figure 10: A comparison between BP and MBP on two more complex synthetic examples. Even if the MBP images are better than BP images and the marginals of MBP are fitted exactly, these results are not really satisfactory.

- For each projection, compute a backprojected image, and in place of adding them up, just multiply them poinwise.

In the next figures, we see some examples.

6 Conclusions

The main contribution of this paper is to find a link between the notion of *copulas* in statistics and X-ray CT. For this, first we presented briefly the bivariate copulas and the image reconstruction problem in CT. We could make a link between the two problems of i) determining a joint bivariate pdf from its two marginals and ii) the CT image reconstruction from only two horizontal and vertical projections, by emphasizing that in both cases, we have the same in-

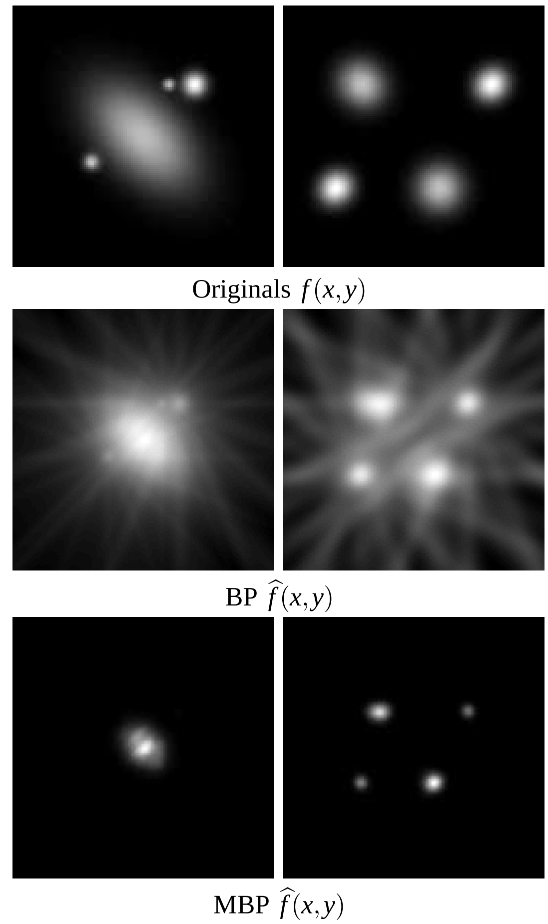


Figure 11: A comparison between BP and MBP on two synthetic examples. Here, we have 08 projections.

verse problem of determining a bivariate function (an image) from the line integrals.

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